The medial axis transform, generalised and combinatorial maps

GEO1004:
3D modelling of the built environment

https://3d.bk.tudelft.nl/courses/geo1004

3D geoinformation
Department of Urbanism
Faculty of Architecture and the Built Environment
Delft University of Technology

Hw1 and pseudonyms form

- Started marking today, should be done tomorrow
- In order to publish your marks, please submit a pseudonym:
  - https://tudelft3d.typeform.com/to/VQIr59aC

Midterm exam info

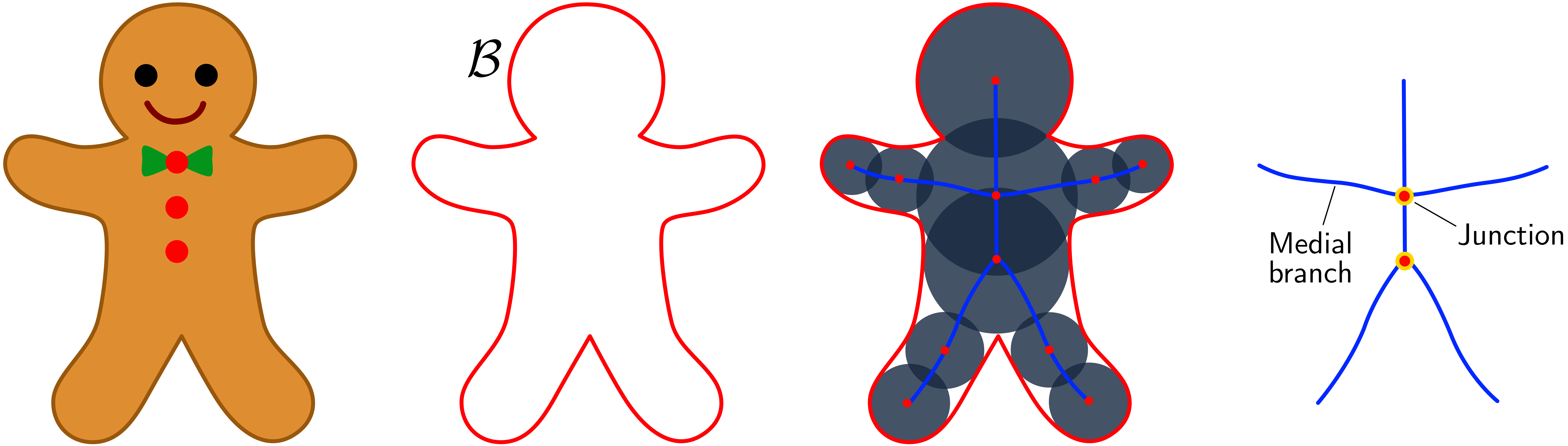
- Next Monday (May 18), 13:45-14:45 (1 hour), 6 open questions
- Lessons 1.1 - 4.1, equivalent to Chapters 1-4 and 7-9
- Open book, open laptop but static materials only (no LLMs!)
- Not allowed to communicate with others, no phone
- https://3d.bk.tudelft.nl/courses/geo1004/midterm/

Thesis intro session

- Next Monday May 18th 16:00-17:00
- How to choose a topic?
- All the rules, timeline, steps (As)
- The graduation system (MyCase)

Medial axis transform (MAT)

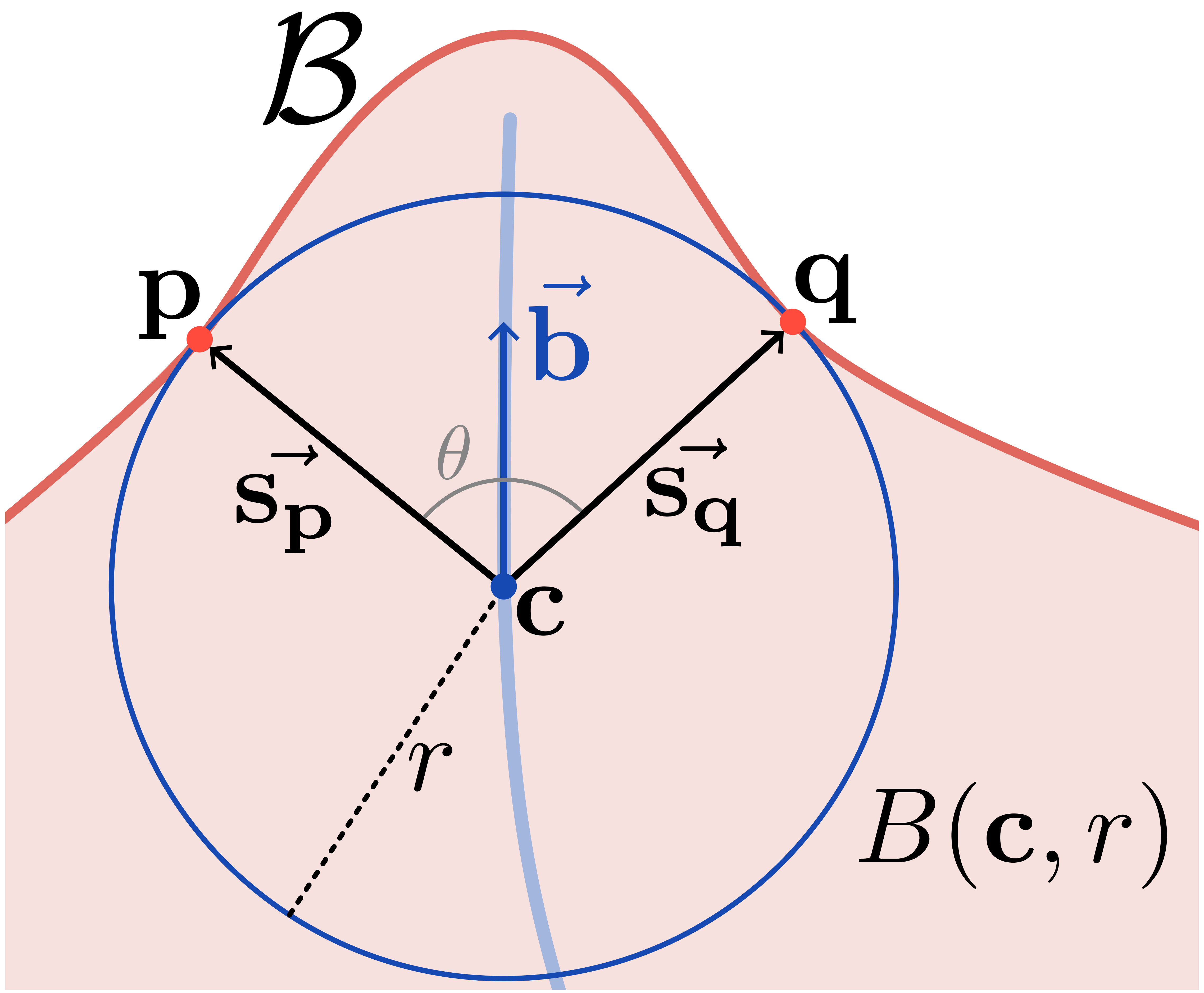
What is the MAT?



Why?

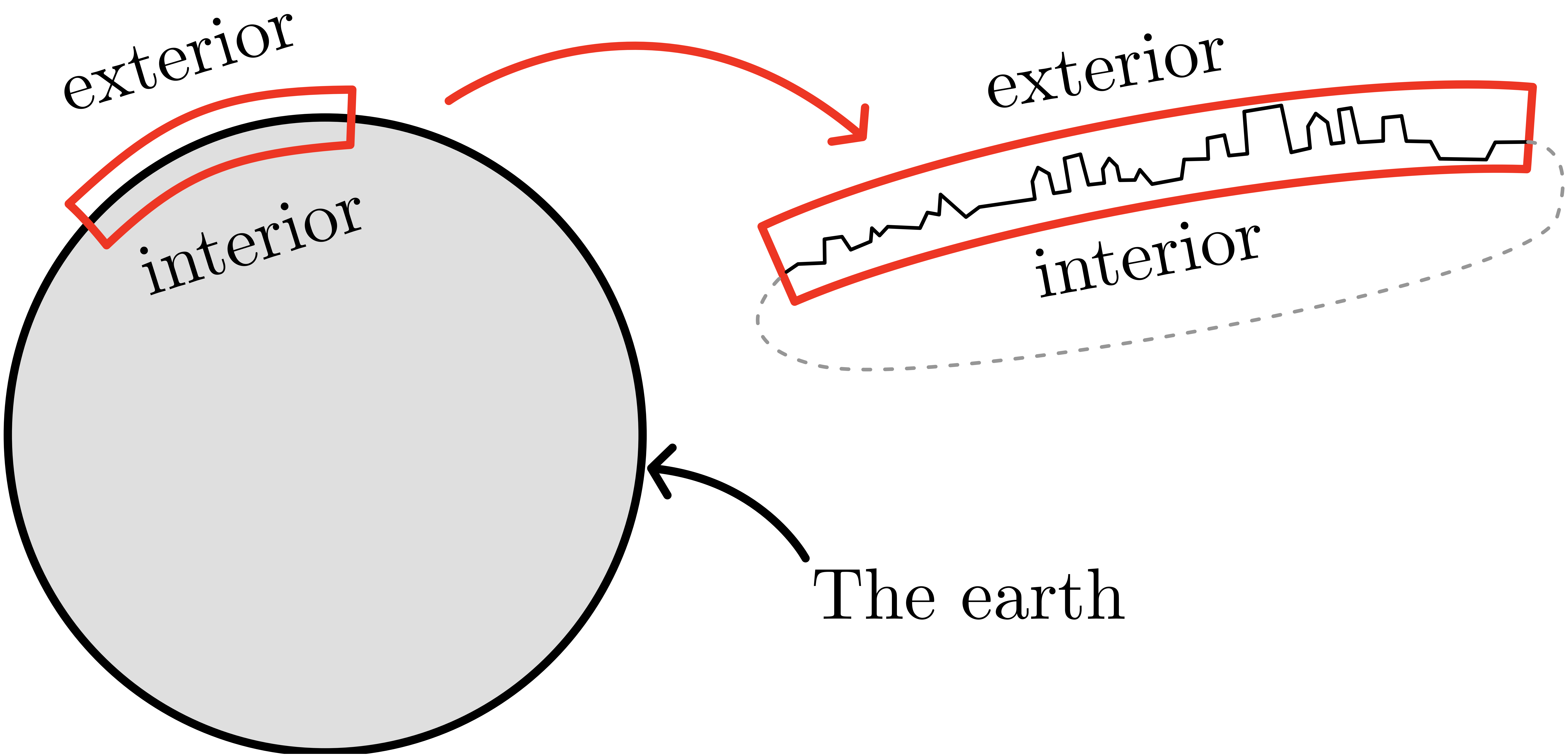
- More natural representation for some questions: branches, thickness, length, etc.
  - In BK: how many hallways? how wide? how long?
- Data processing method:
  - Dimensionality reduction (like b-rep): from areas to lines, from volumes to areas
  - Extraction of features, segmentation, noise removal, etc.

MAT (parts)

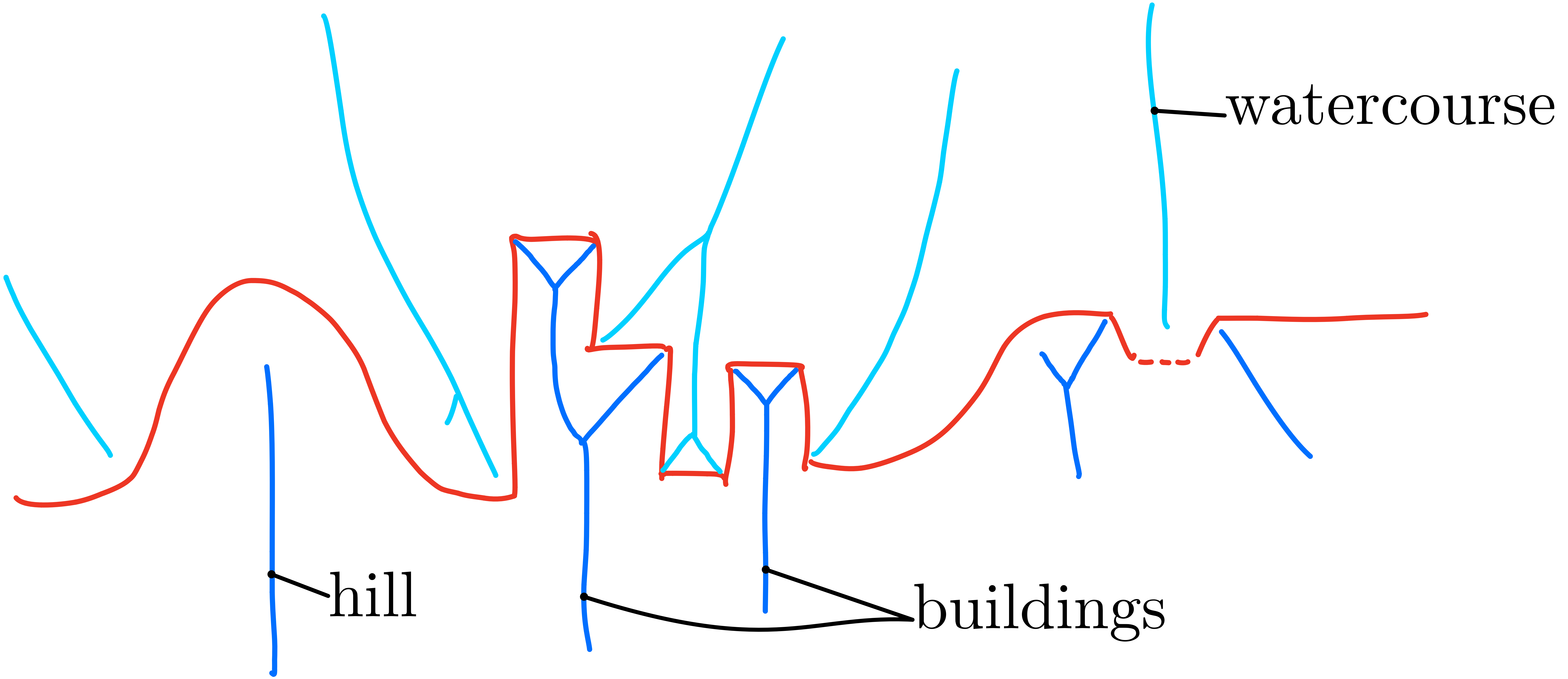


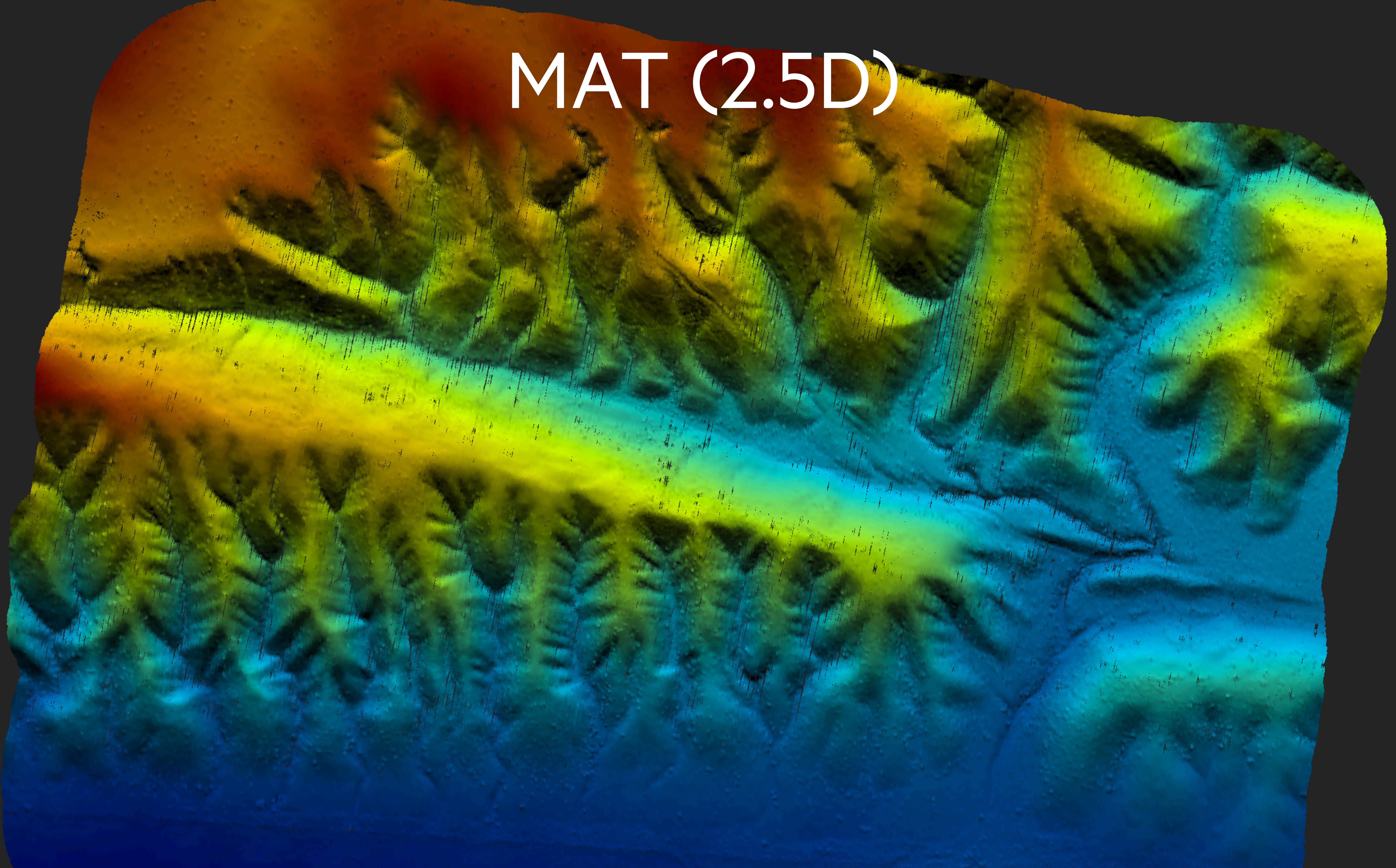
Symbol	Description
$B(\mathbf{c}, r)$	medial ball
$\mathbf{c}$	medial atom
r	radius
$\mathbf{p}, \mathbf{q}$	feature points
$\vec{\mathbf{s}}_\mathbf{p}, \vec{\mathbf{s}}_\mathbf{q}$	spoke vectors
θ	separation angle
$\vec{\mathbf{b}}$	medial bisector

MAT (vertical section / 2.5D)



MAT (vertical section / 2.5D)




MAT (2.5D)

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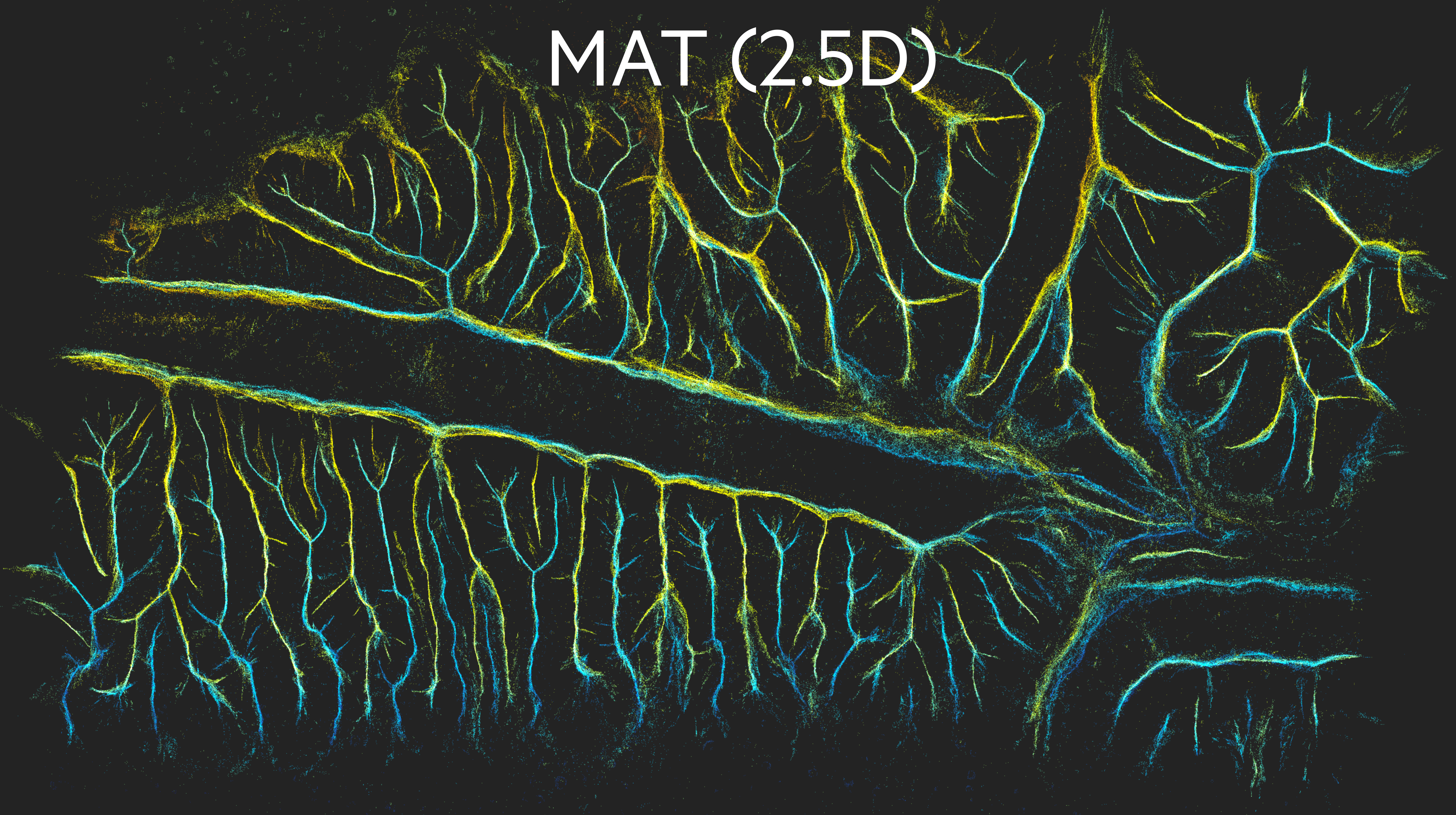
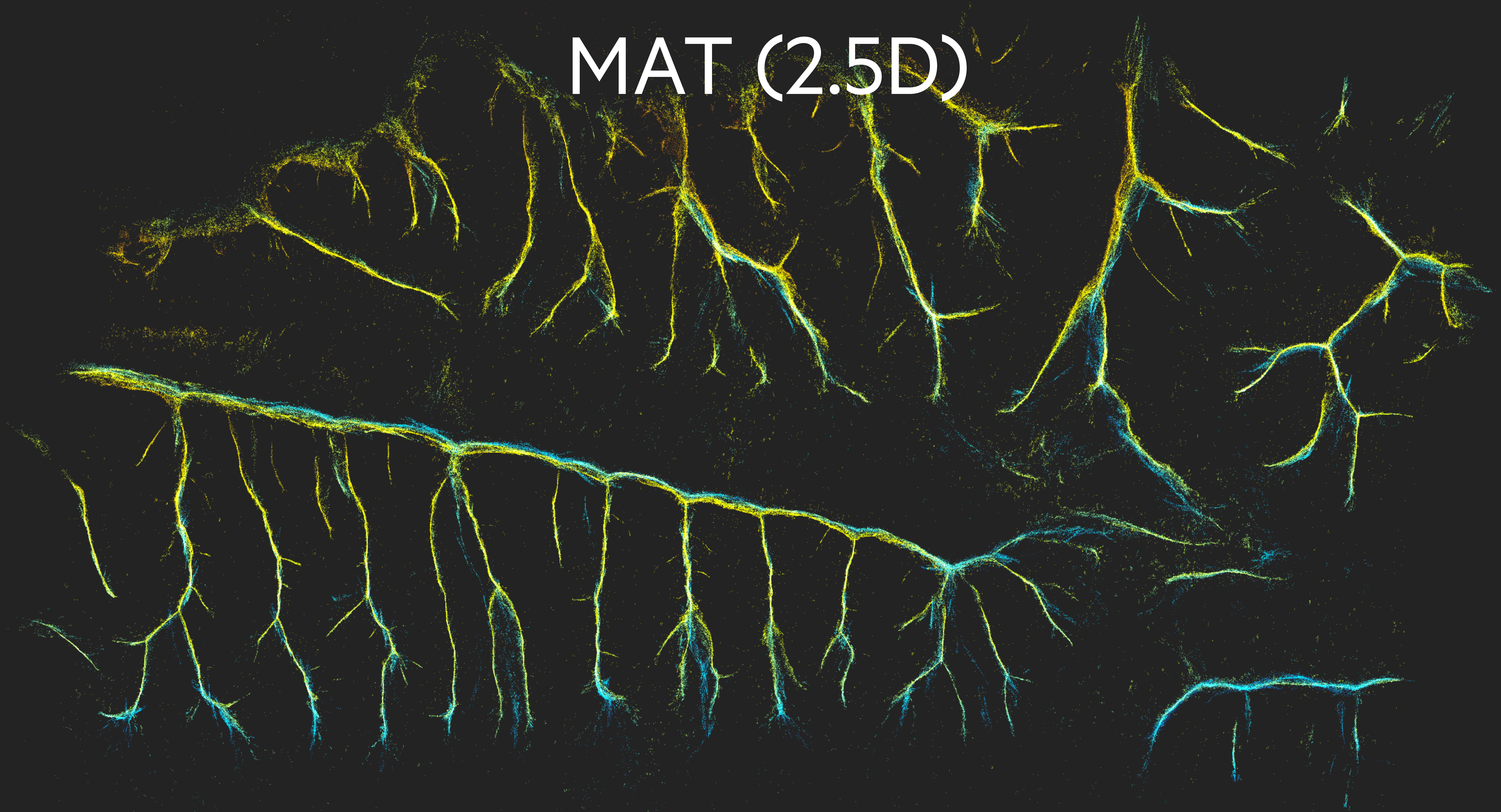
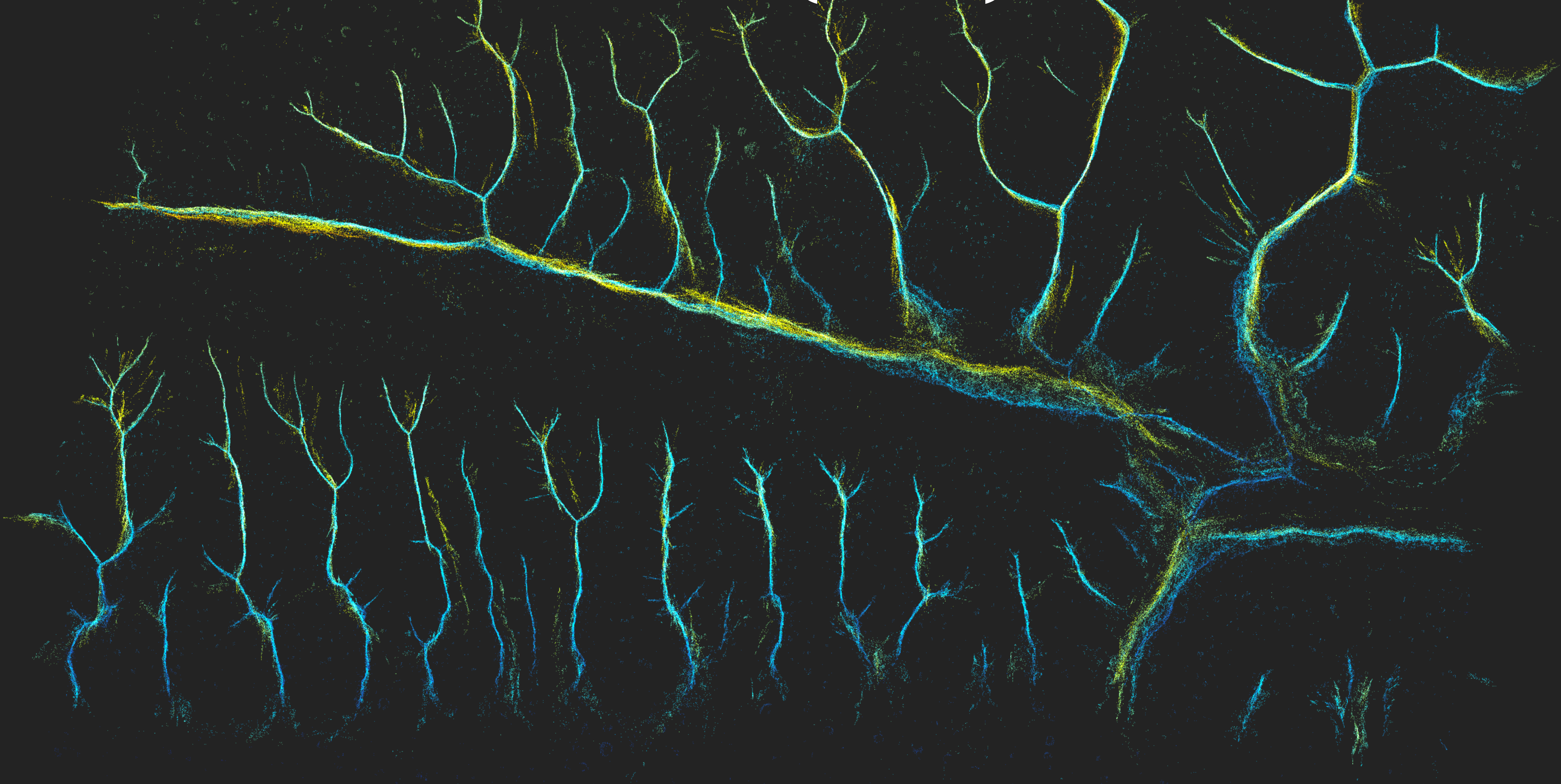


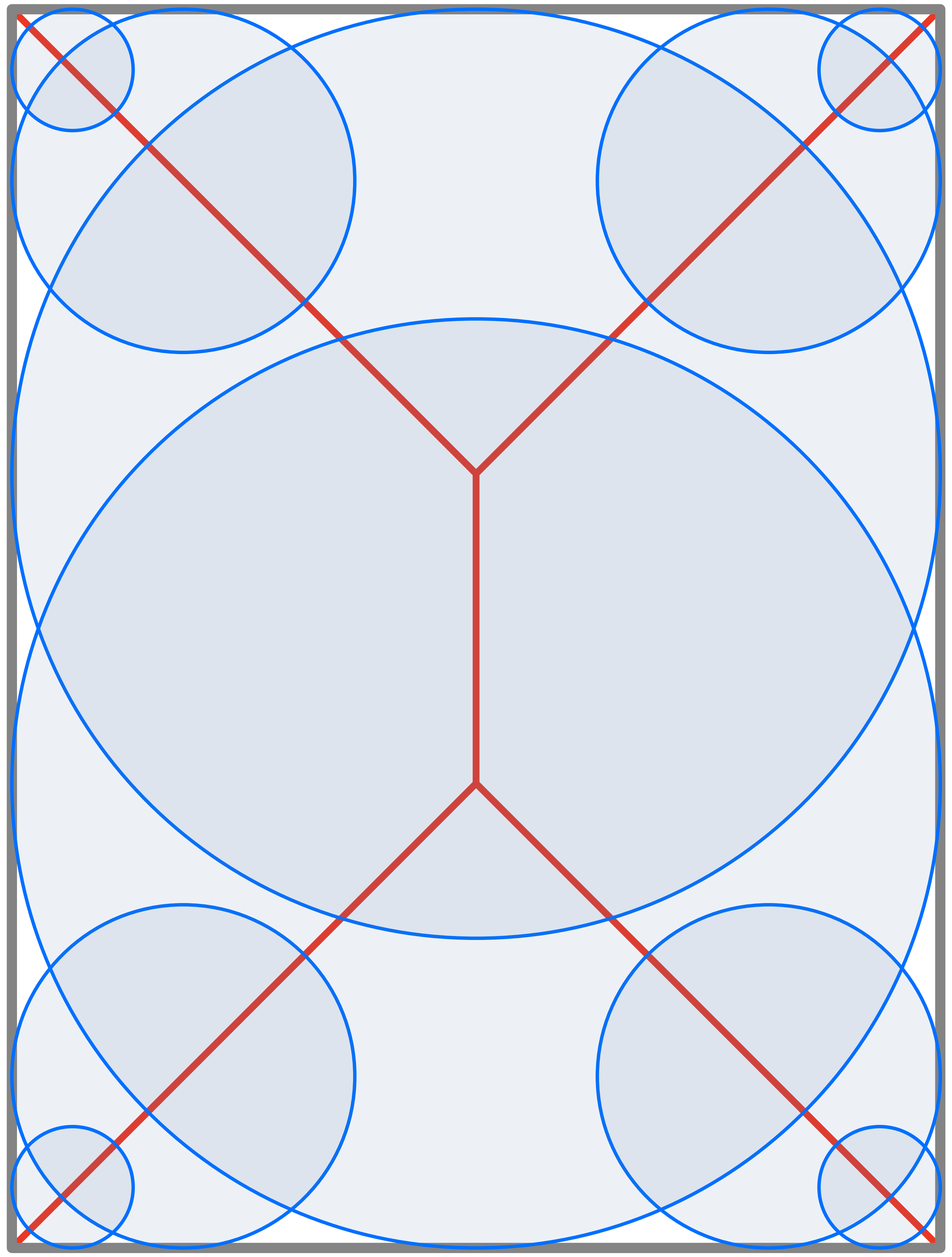
MAT (2.5D)



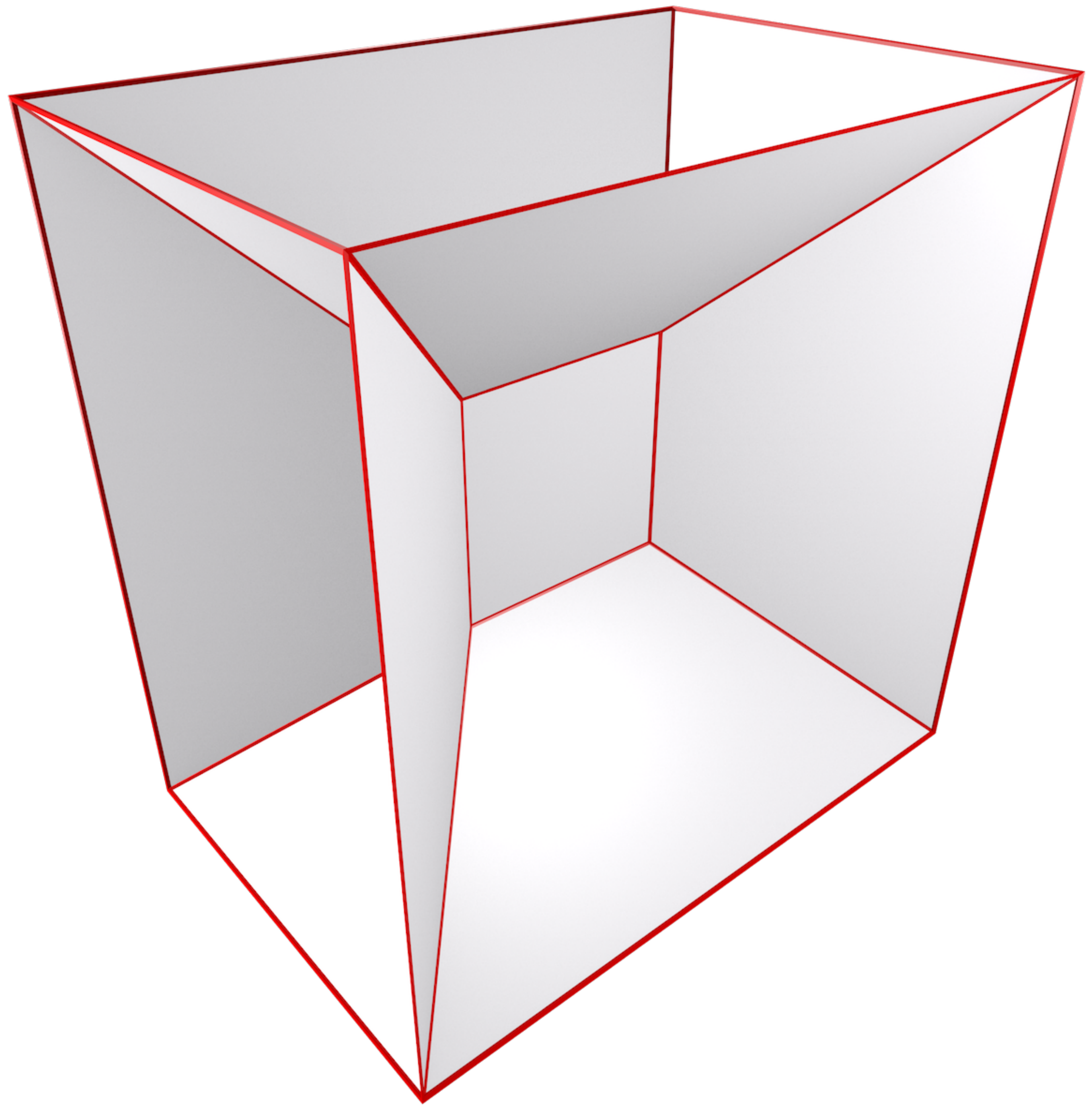
MAT (2.5D)

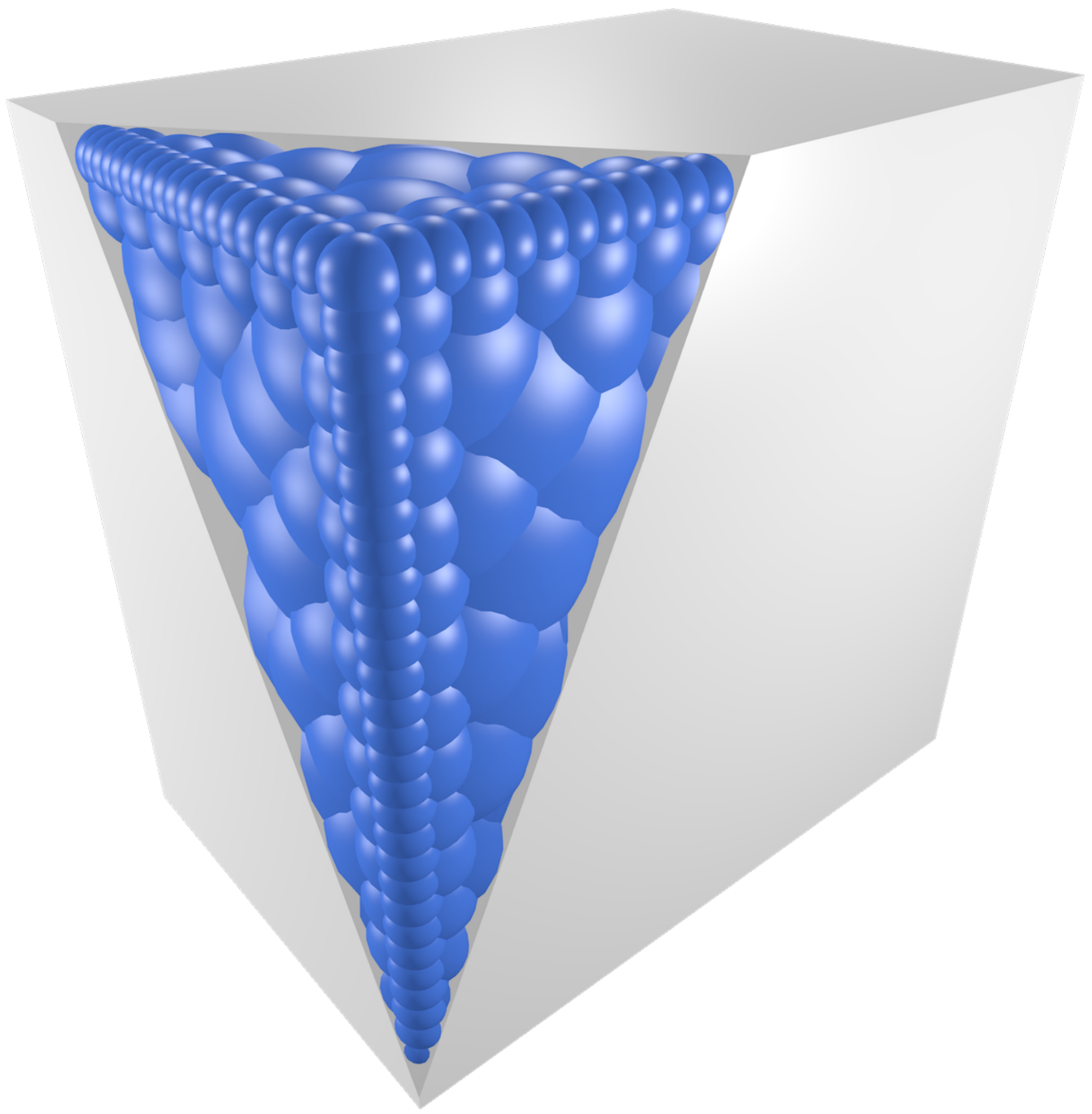


MAT (3D)

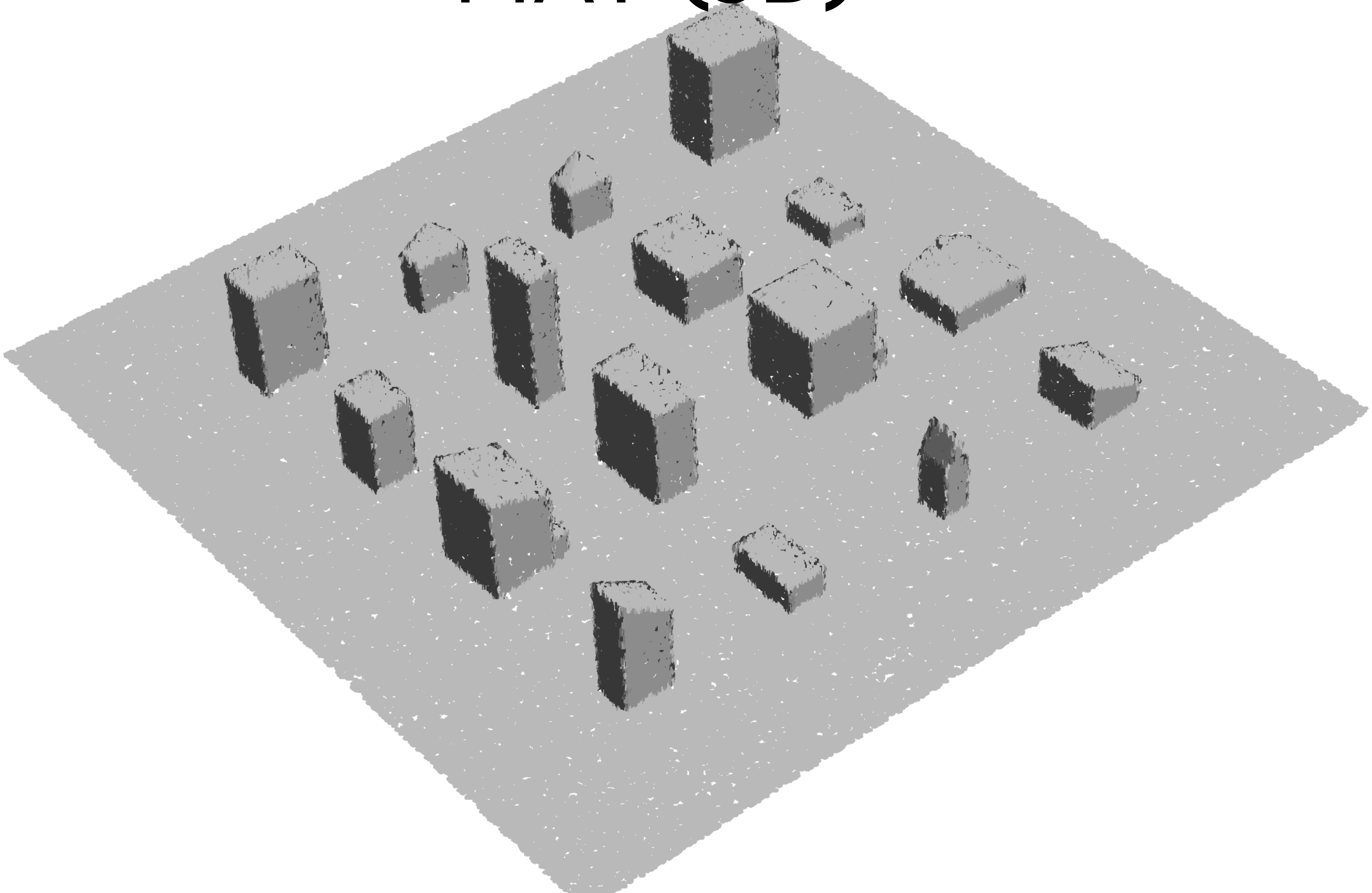


MAT (3D)





MAT (3D)

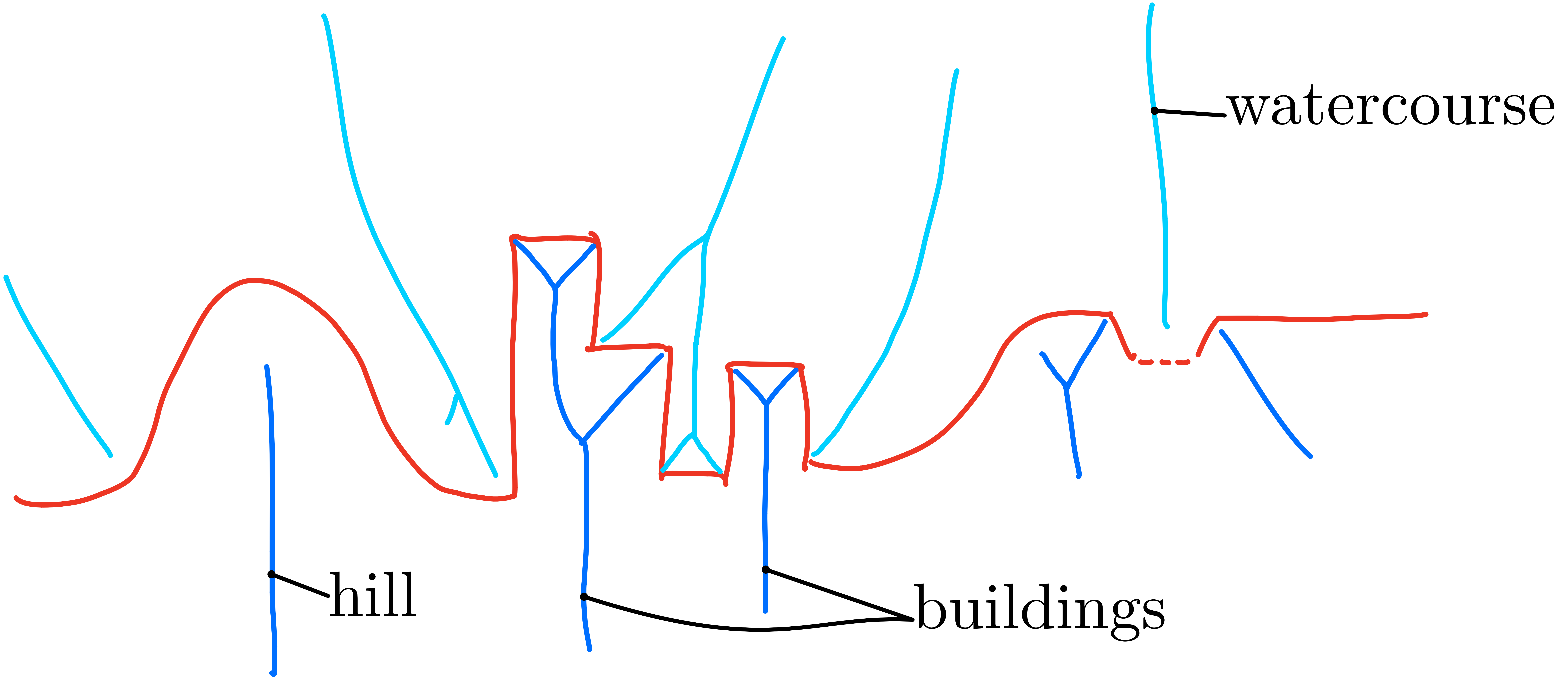


point cloud

MAT (3D)

MAT from PC

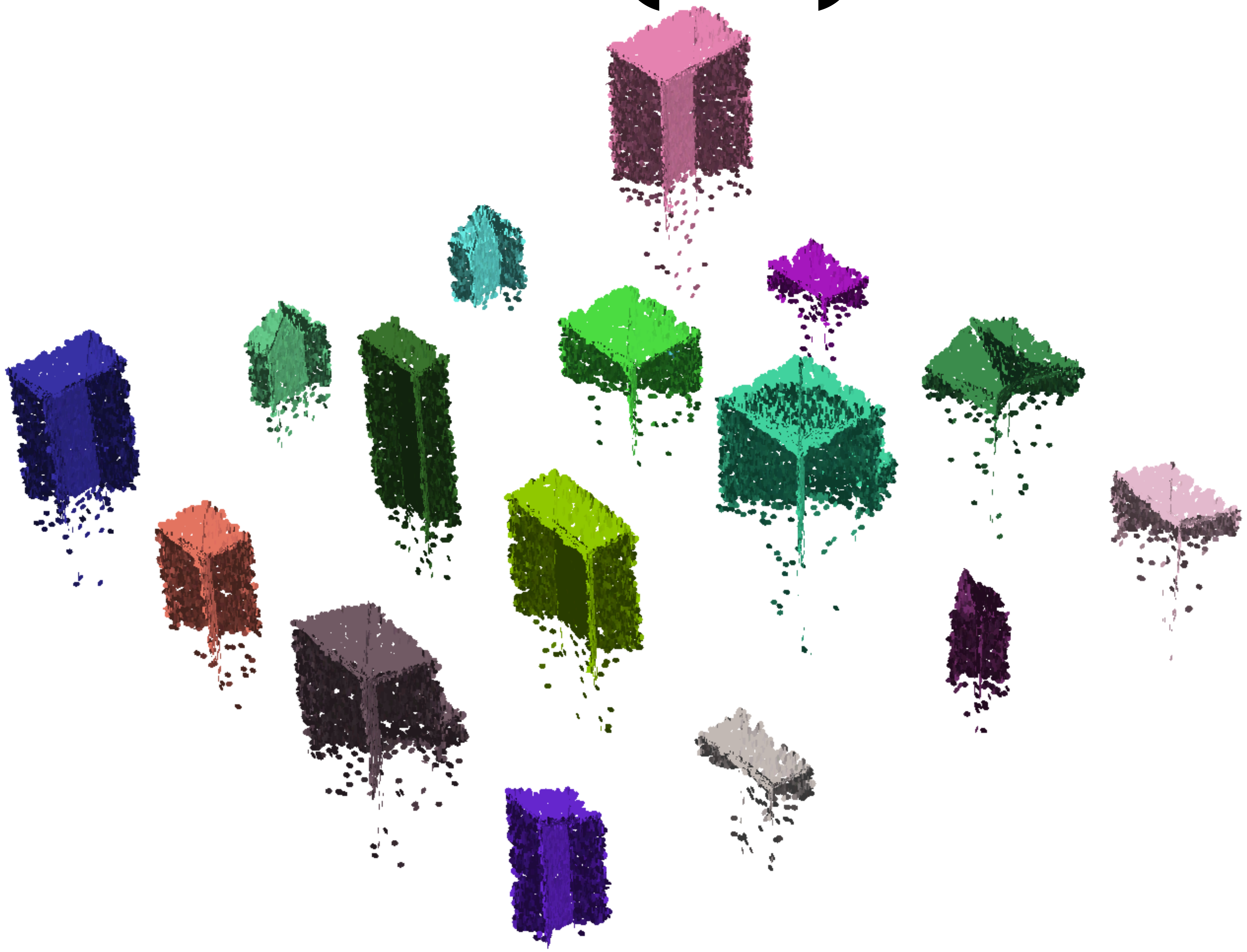
MAT (vertical section / 2.5D)



MAT (3D)

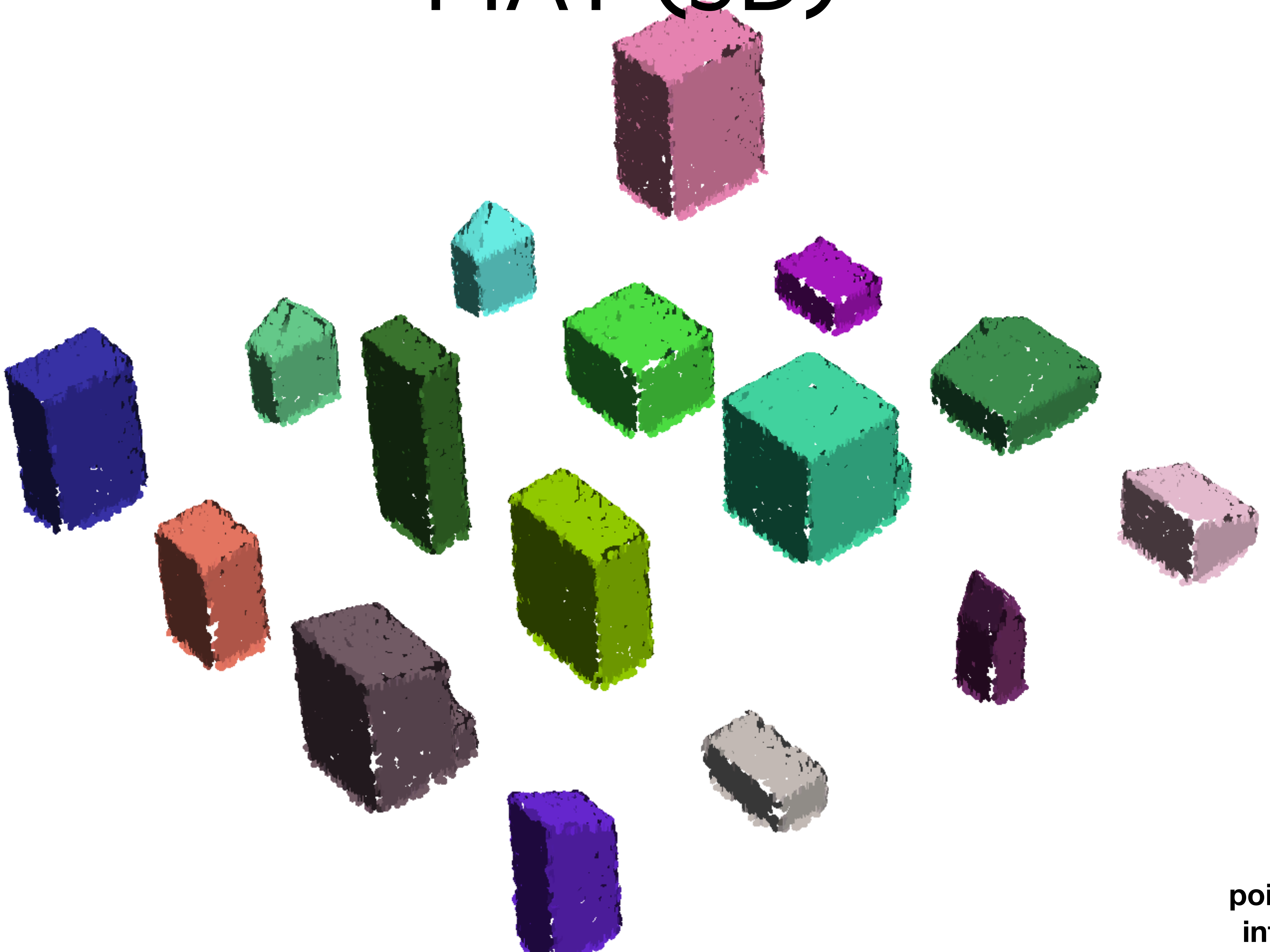
medial clusters

MAT (3D)



interior medial clusters

MAT (3D)



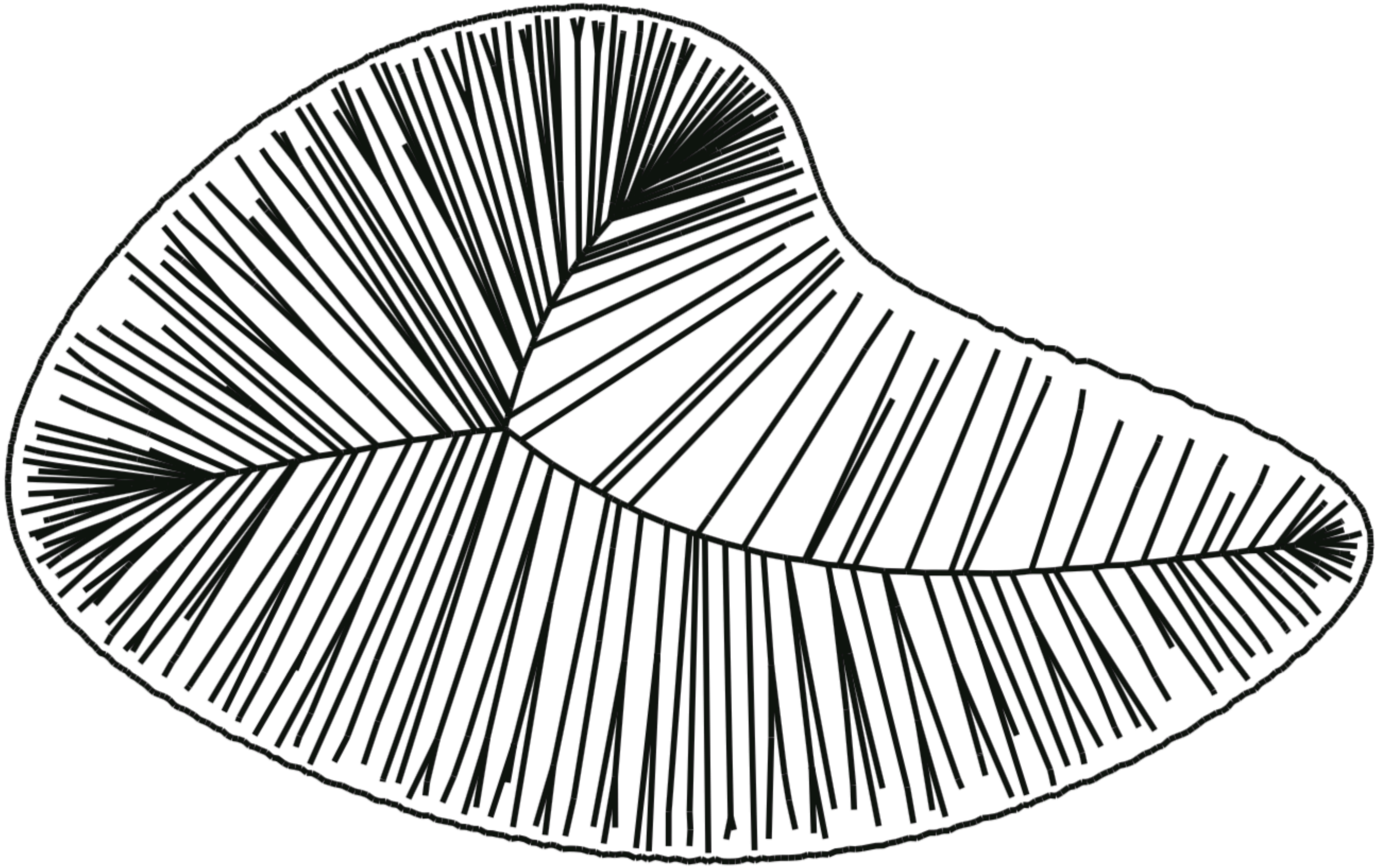
points corresponding to interior medial clusters

How to compute the MAT?

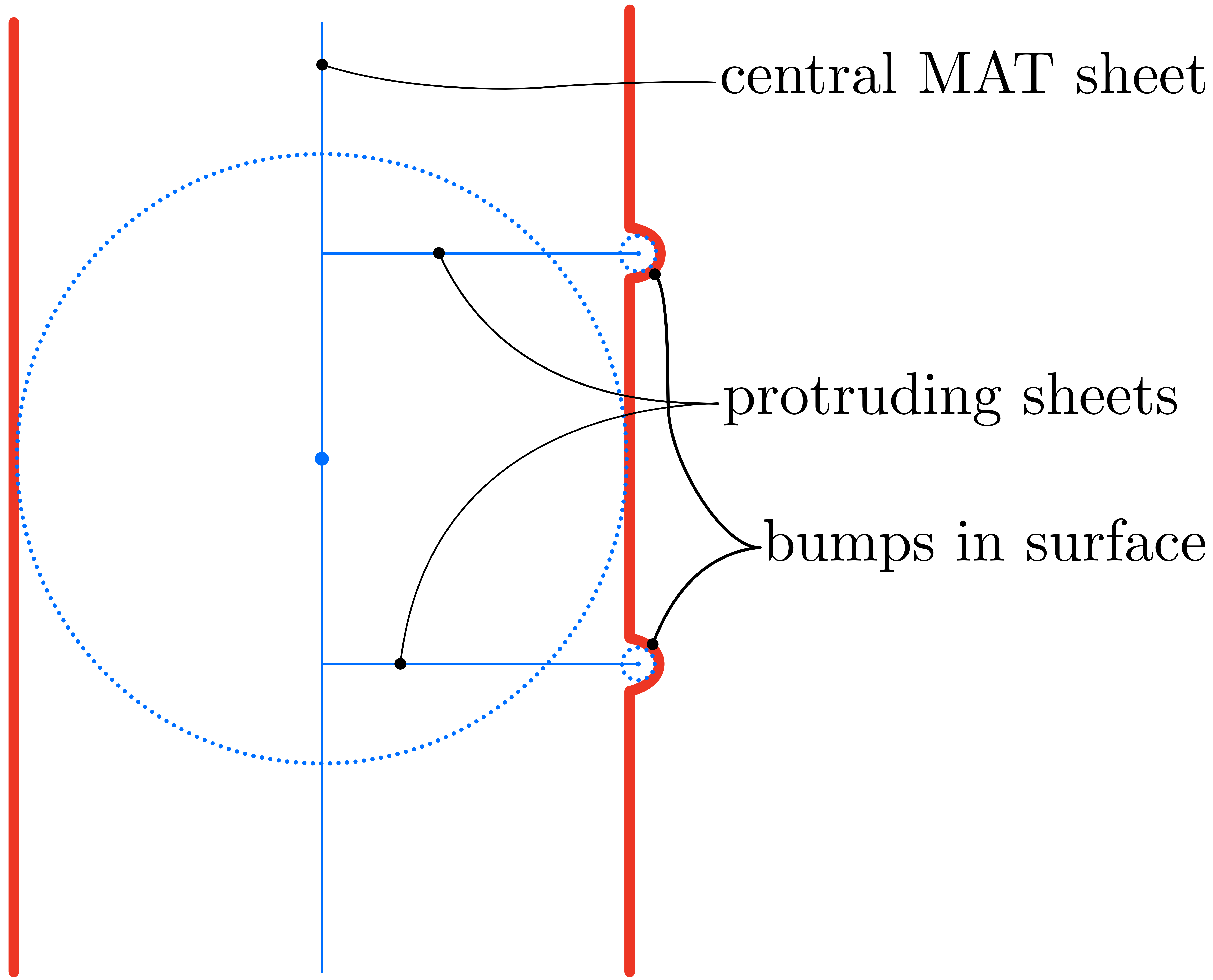
- Some methods:
  - voxelisation, then iterative removal of boundary voxels (thinning)
  - voxelisation, then computing distance to exterior (distance transform), then finding ridges
  - sampling of boundary, then Voronoi diagram, then selection of cells inside the original shape
  - shrinking ball algorithm directly from PC with normals

Shrinking ball algorithm

Pruning



Pruning

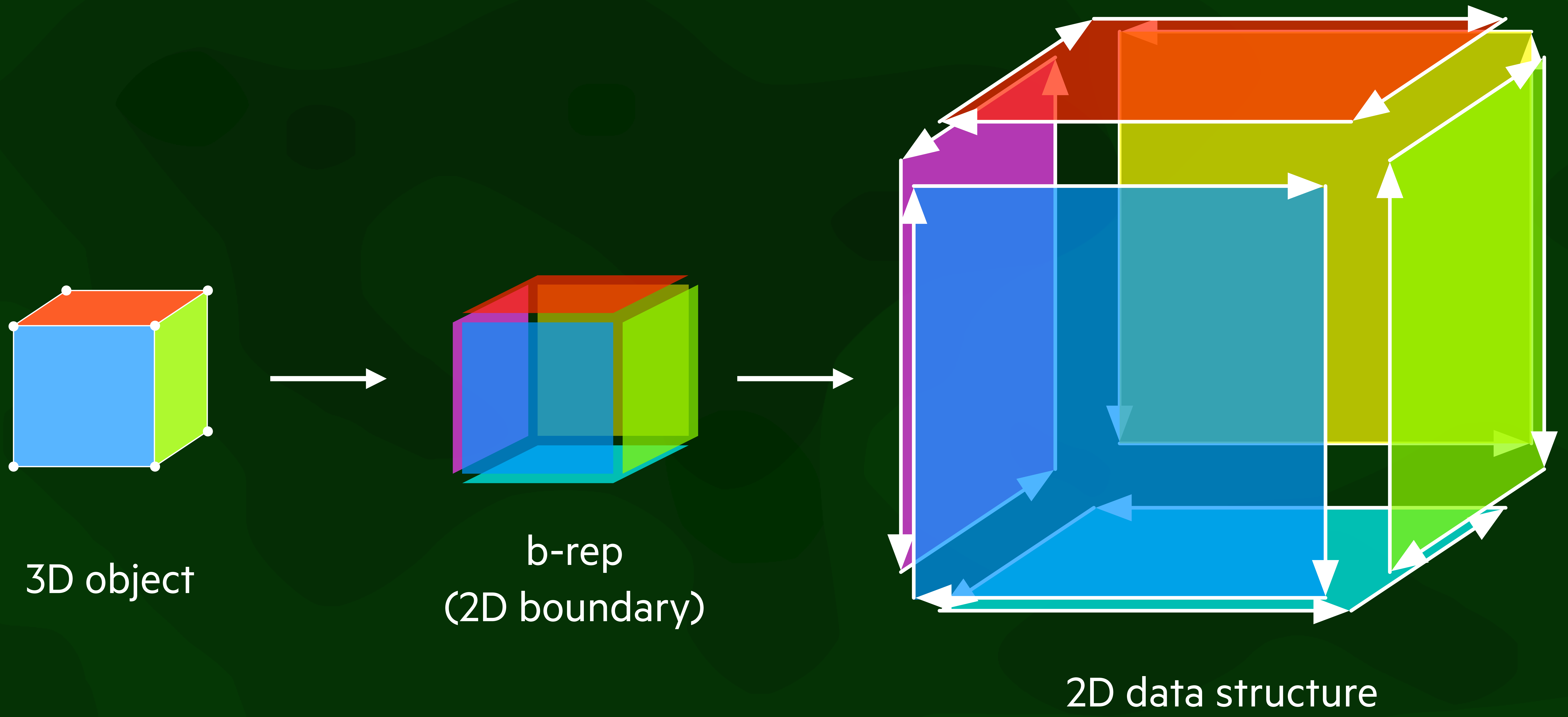


Pruning

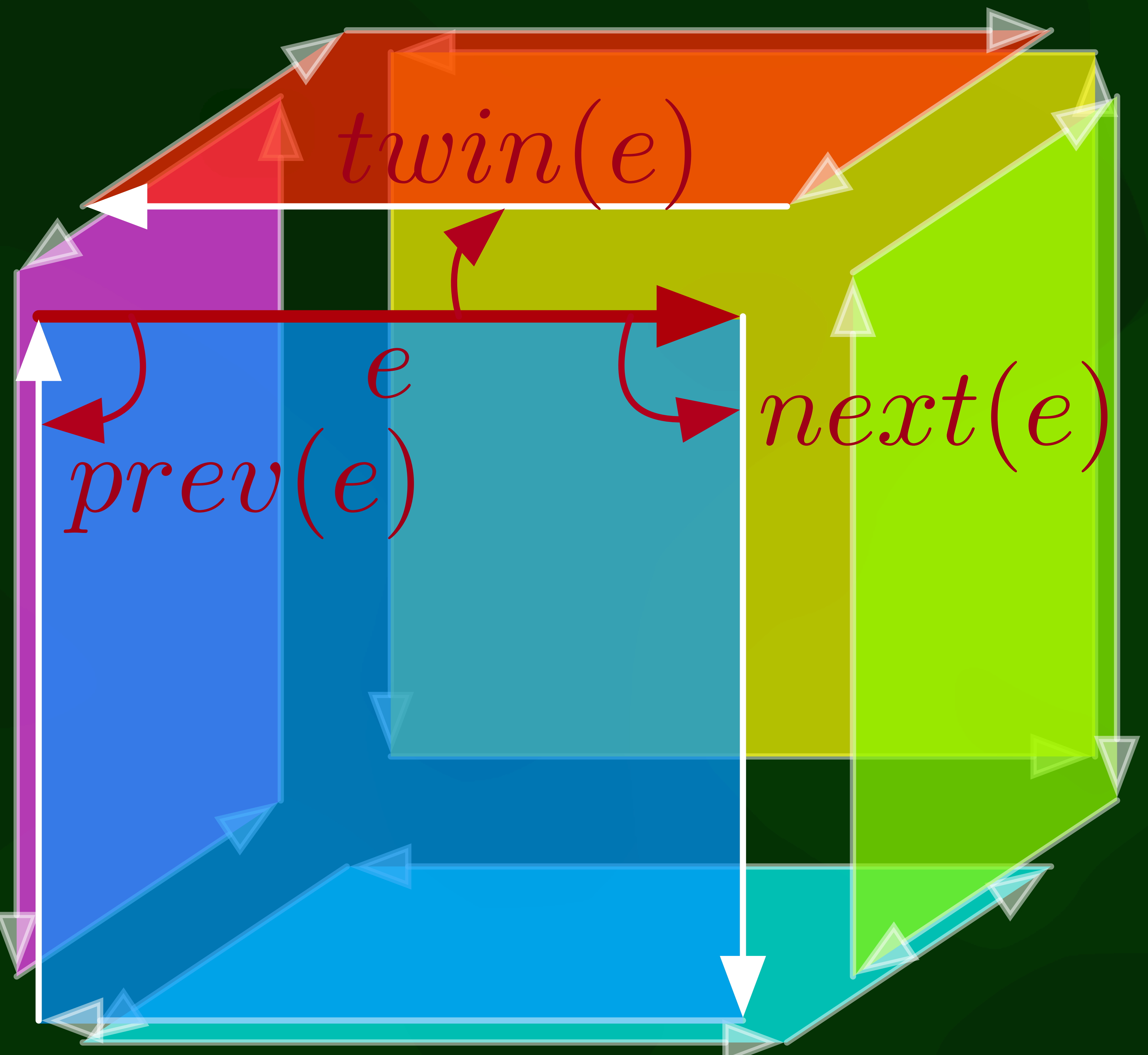

0
50
100
150
200
250
300
0
100
200
300
400
500
600
700
0
50
100
150
200
250
300
0
100
200
300
400
500
600
700

Generalised and combinatorial maps

So far… (3D through b-rep)

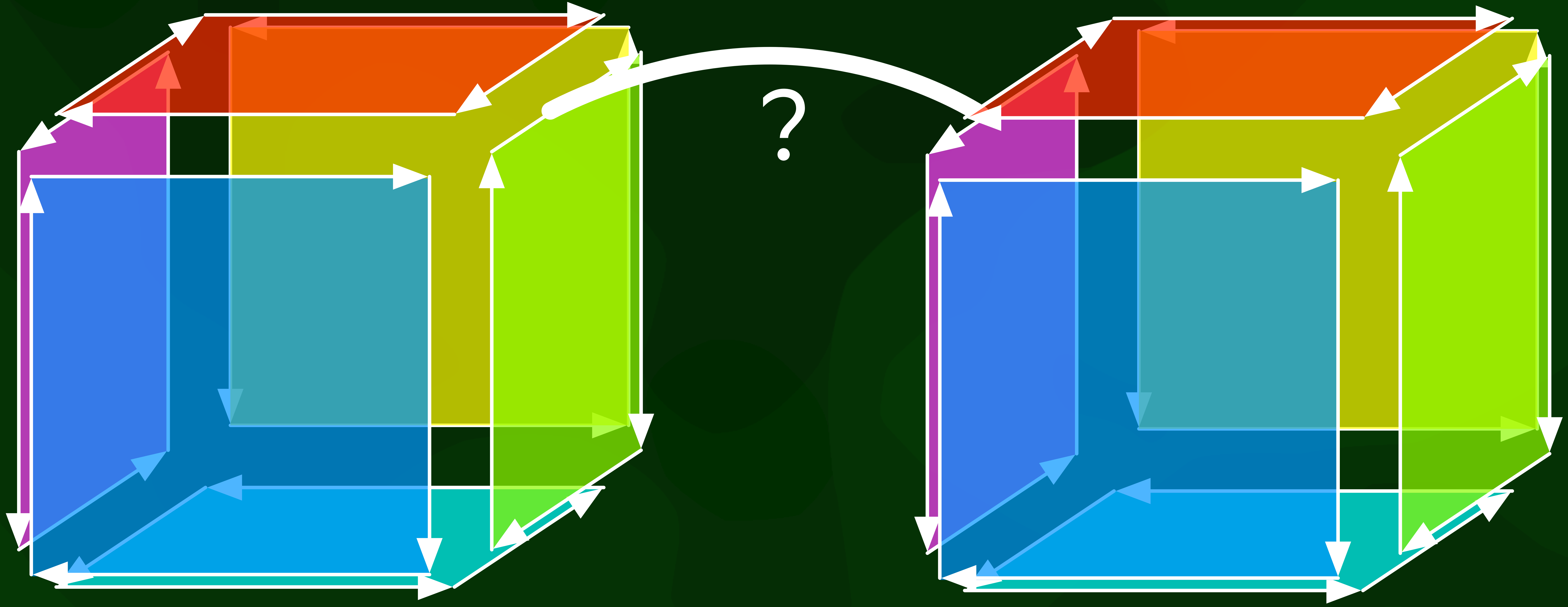


So far... (3D through b-rep)



links between 0D-2D elements

Links between 3D elements?



Drawbacks of b-rep approach

- Difficult to store:
  - Multiple volumes
  - Holes (2D and 3D)
  - Non-manifolds

Back to Jordan-Brouwer theorem

- In 2D, the Jordan curve theorem says: a closed curve separates the plane into two parts: an interior surface and an exterior surface

- In nD, the Jordan-Brouwer theorem, which in 3D says: a closed surface separates 3D space into two parts: an interior volume and an exterior volume.

Back to Jordan-Brouwer theorem (3D)

- Problems:
    - Holes: one more exterior per hole
    - Multiple volumes: one more interior per extra volume
    - Non-2-manifold: possibly one more interior per point in non-manifold part (depending on orientation)
- Note: can be fixed with bridges or by using a particular orientation

… but still links between 3D objects are missing

What are g-maps / c-maps?

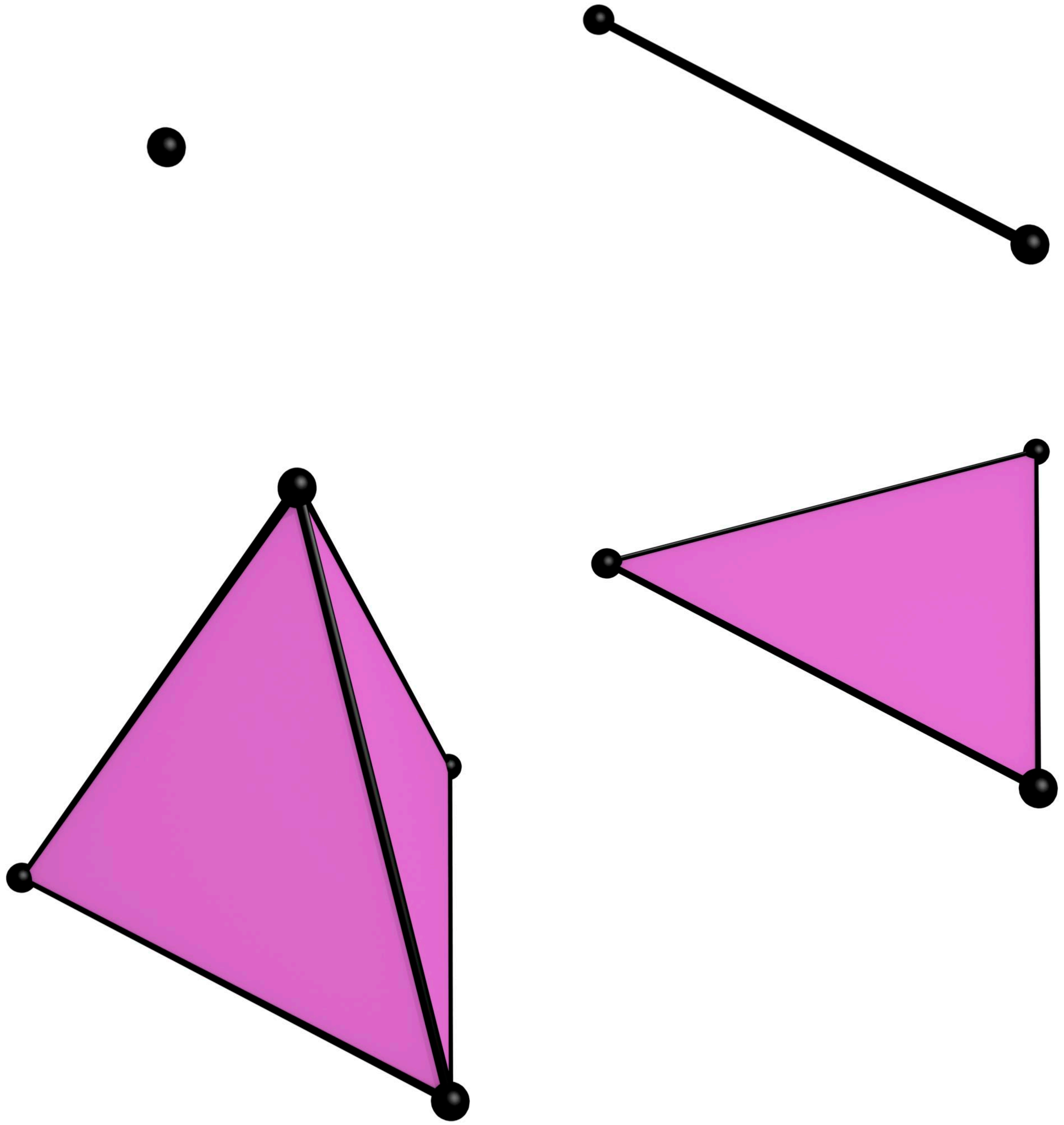
- In short, nD data structures, that is data structures that can store:
  - objects of any dimension
  - and the topological relationships between them
- c-maps: generalisation of half-edge to nD -> 2D c-maps is half-edge
- g-maps: c-maps where each element is split into two to avoid oriented edges

Why g-maps / c-maps?

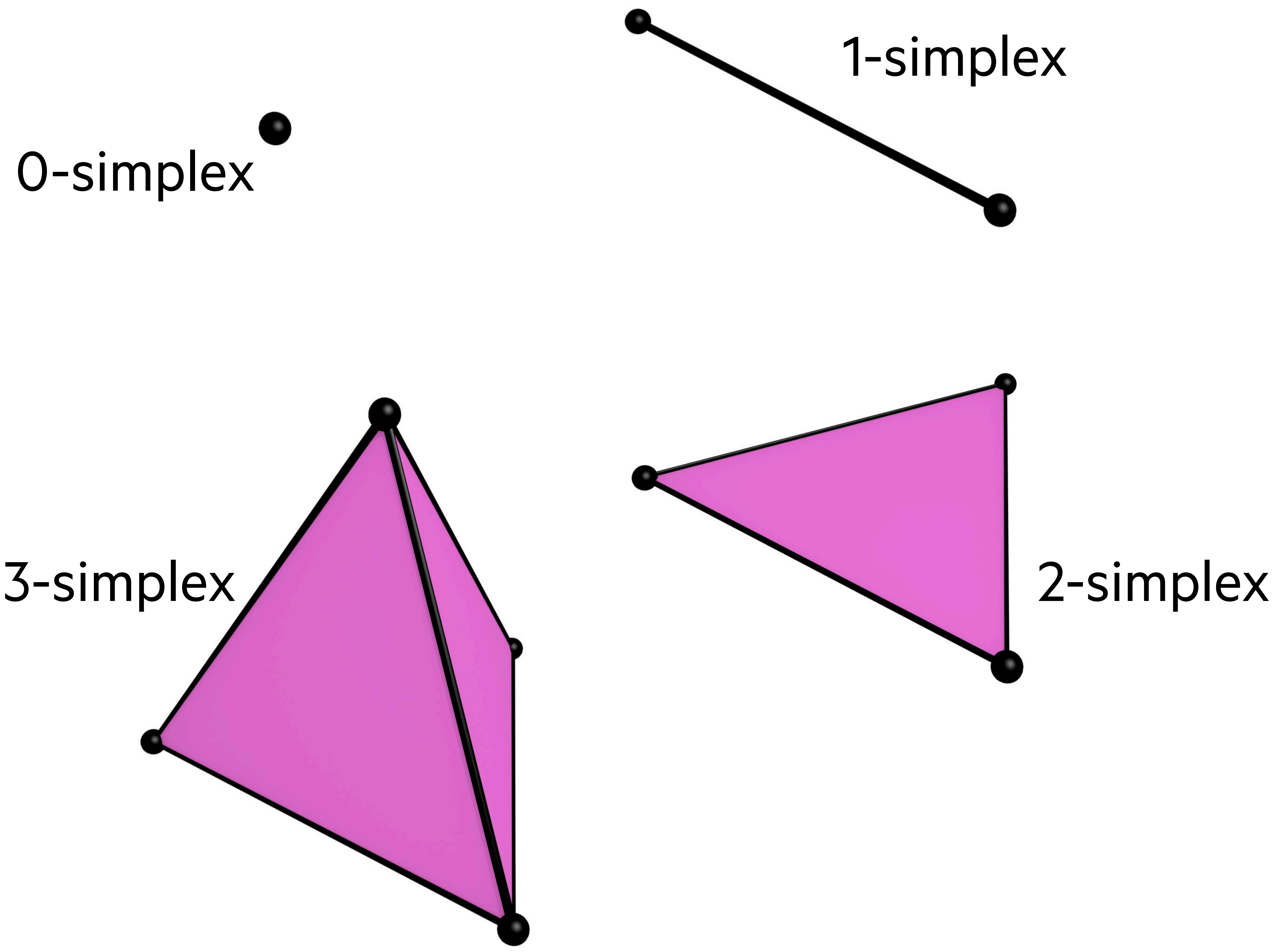
- In short:
  - Possibility to store links between nD elements (including 3D)
  - With g-maps: no orientation issues (e.g. in construction or with dangling/invalid objects)

Small background

Simplex



Simplex



Simplex properties

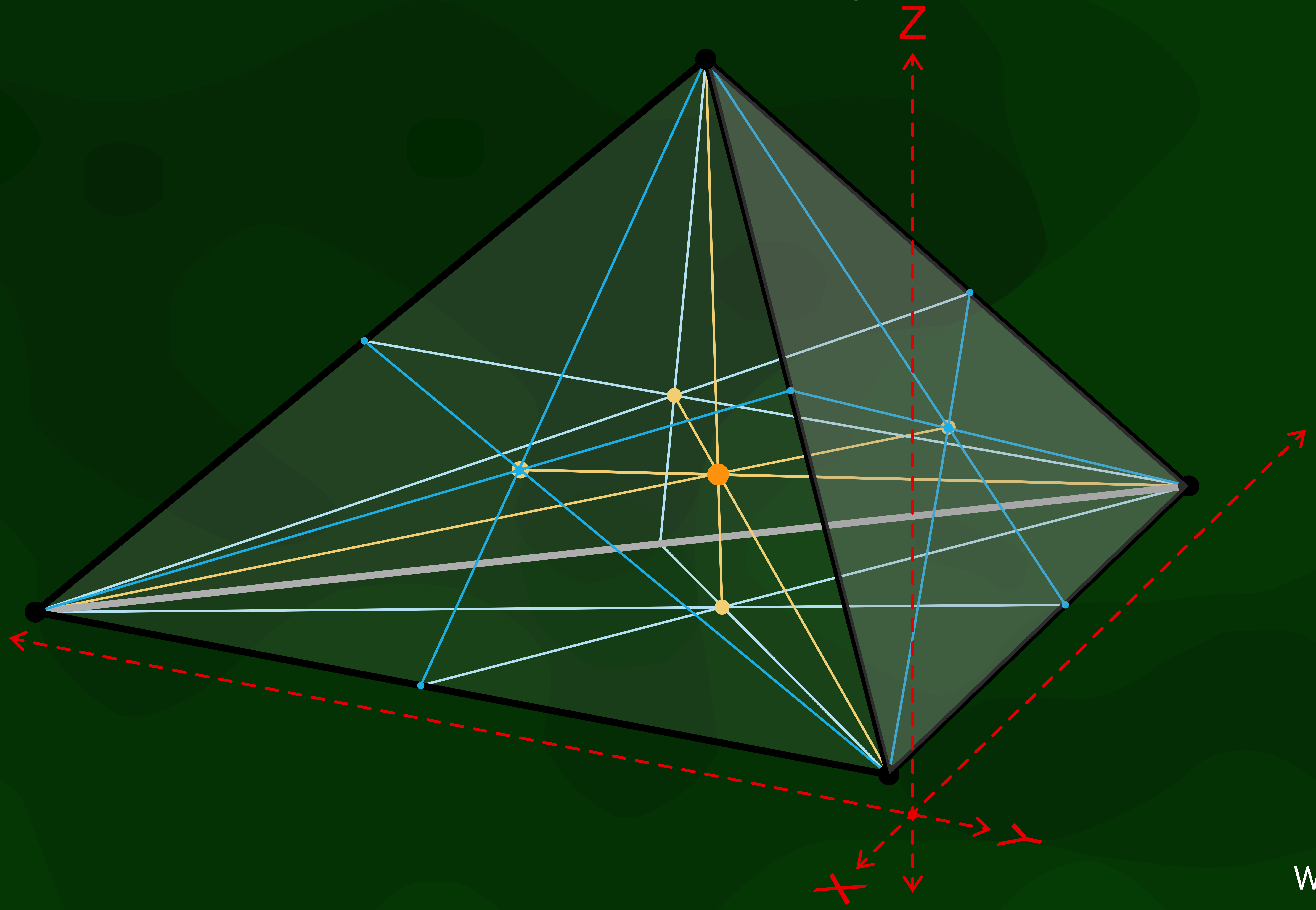
- An n-simplex in n-dimensional space:
  - is bounded by n+1 (n-1)-simplices
  - can have n+1 adjacent n-simplices, each of which shares a (n-1)-simplex on their common boundary

➔ Two adjacent n-simplices share all their vertices except for one

Cells

- 0-cell: vertex
- 1-cell: edge
- 2-cell: polygon
- 3-cell: polyhedron
- …

Barycentric triangulation

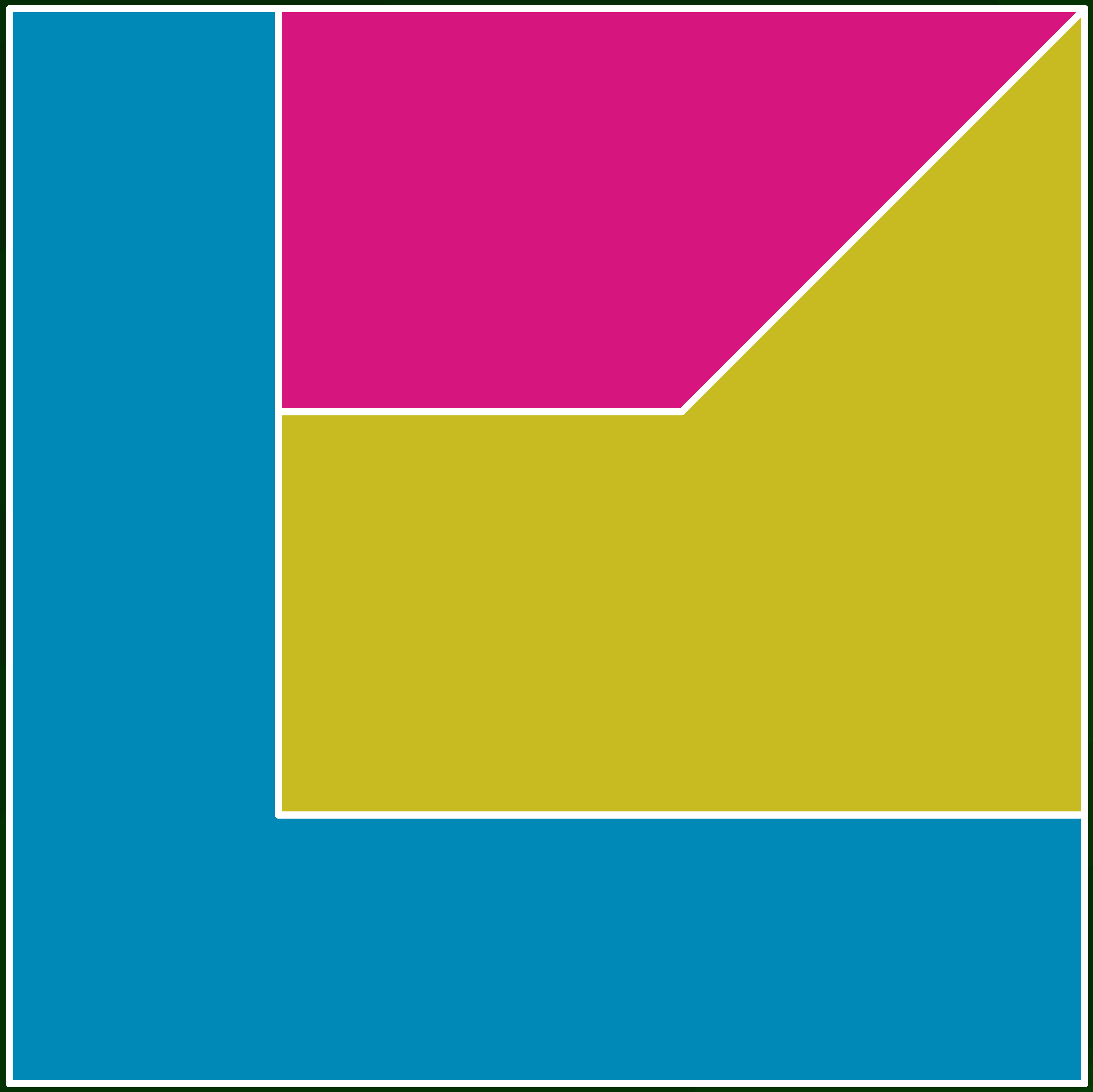


Wikimedia Commons

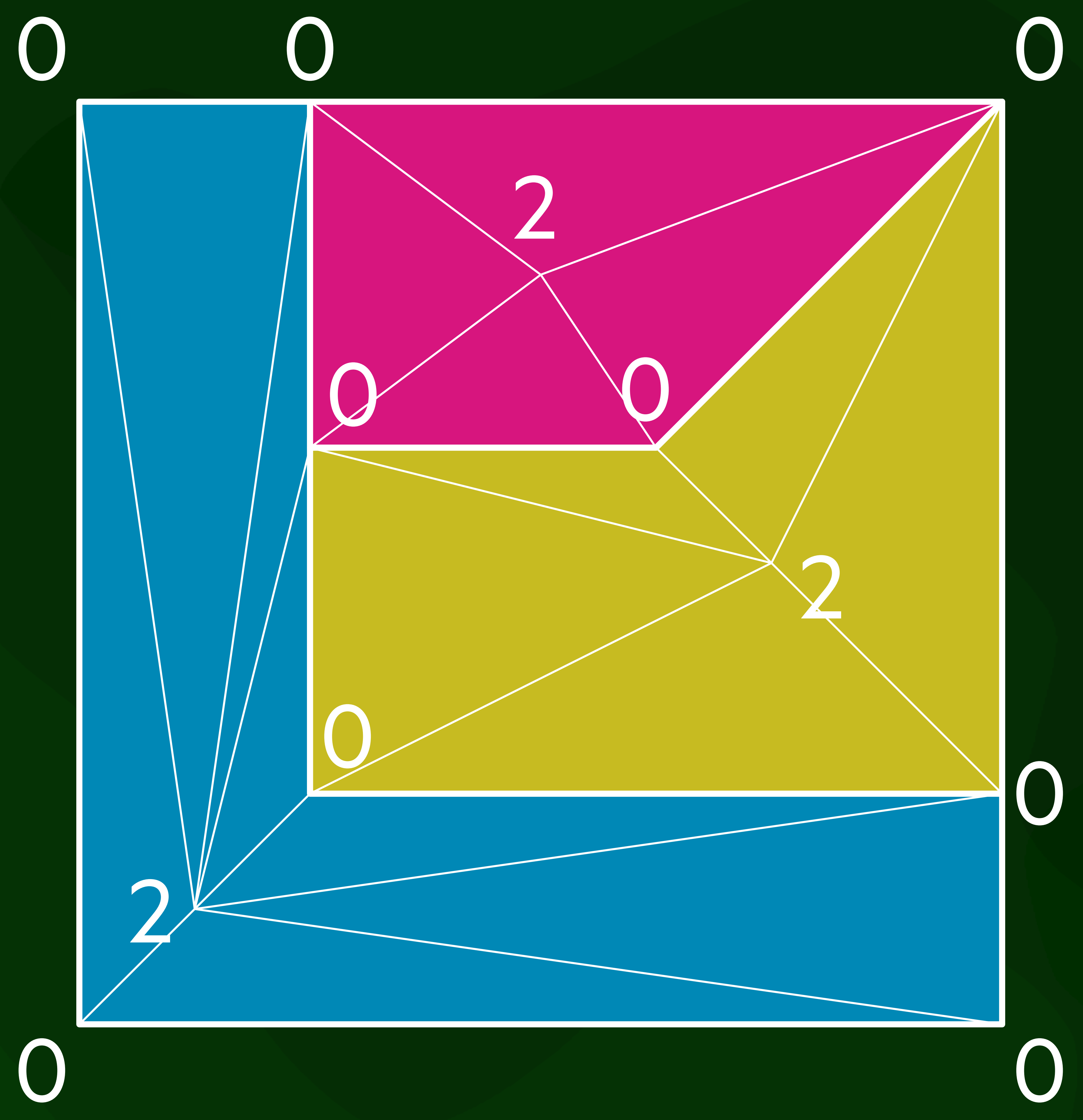
Building g-maps / c-maps

- nD c-maps (n-c-maps): barycentric triangulation n-cells from 2-cells
- nD g-maps (n-g-maps): barycentric triangulation of n-cells from 1-cells
- … where
  - n-simplices are called **darts** and have vertices that are linked to elements of a certain dimension, and
- when only 0-cells have a location in space and linear geometries are assumed between them, it is known as a **linear cell complex**.

Building g-maps / c-maps

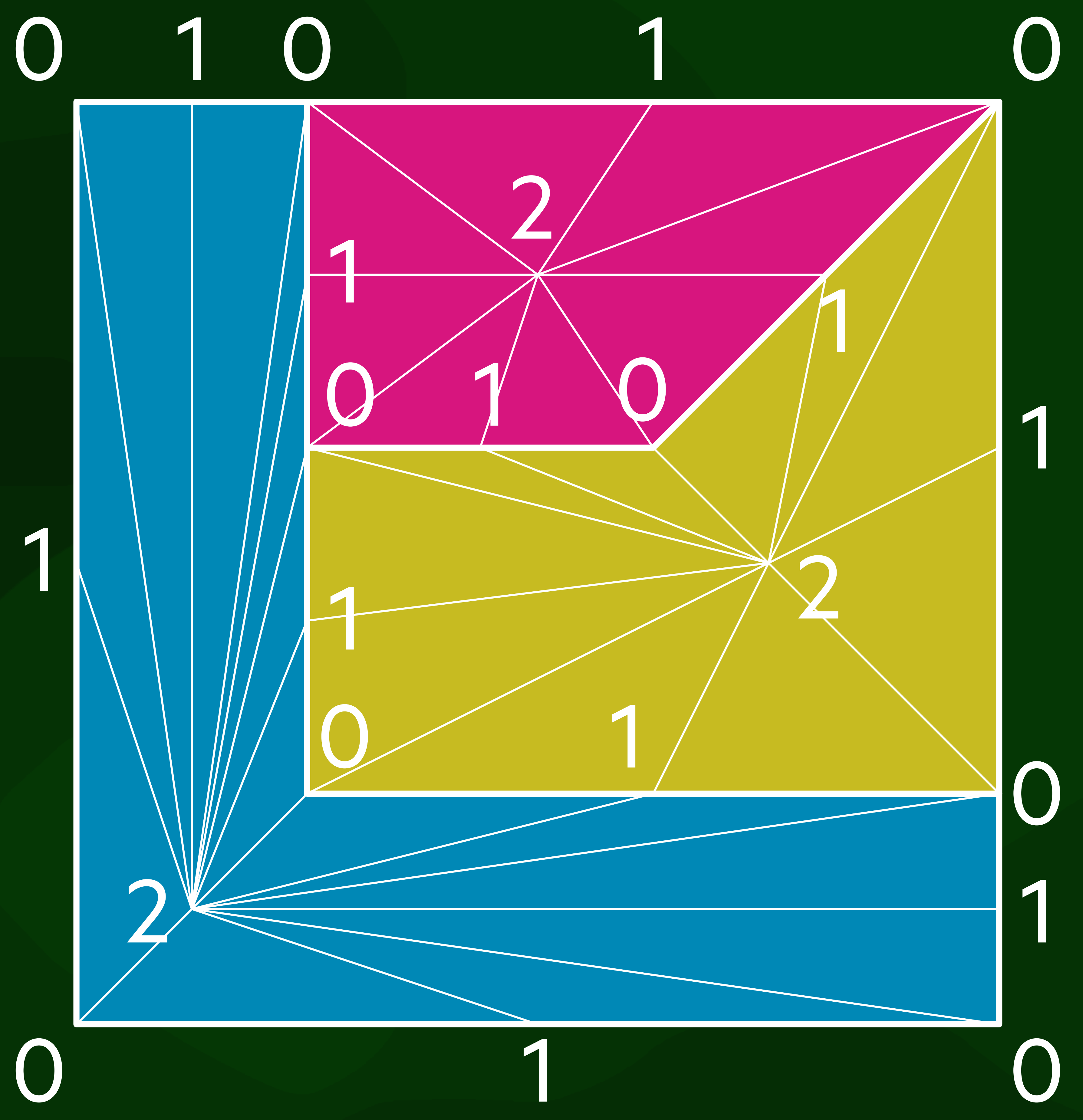


Building g-maps / c-maps



combinatorial map

(c-map)



generalised map

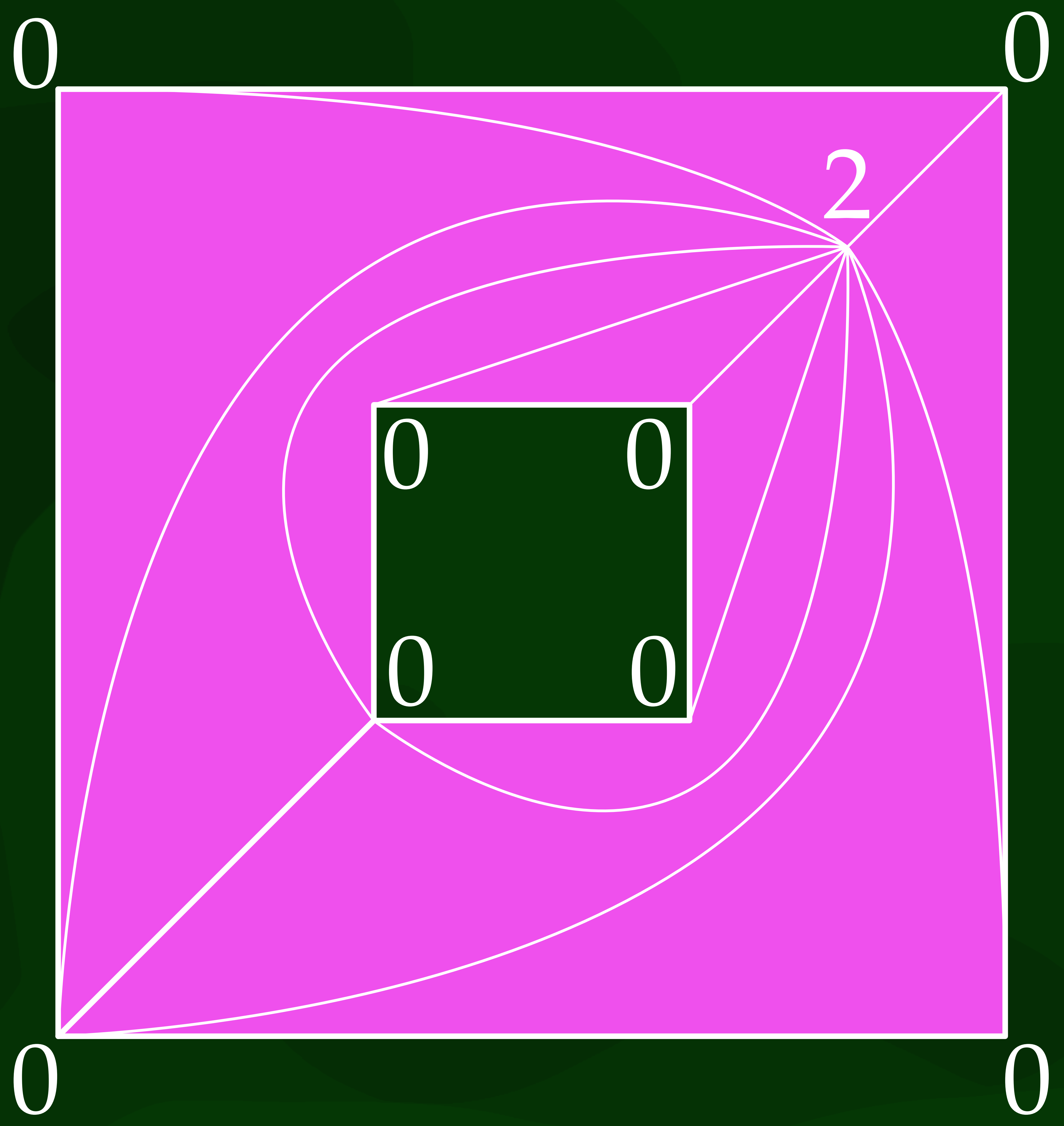
(g-map)

Intuitive meaning of a dart

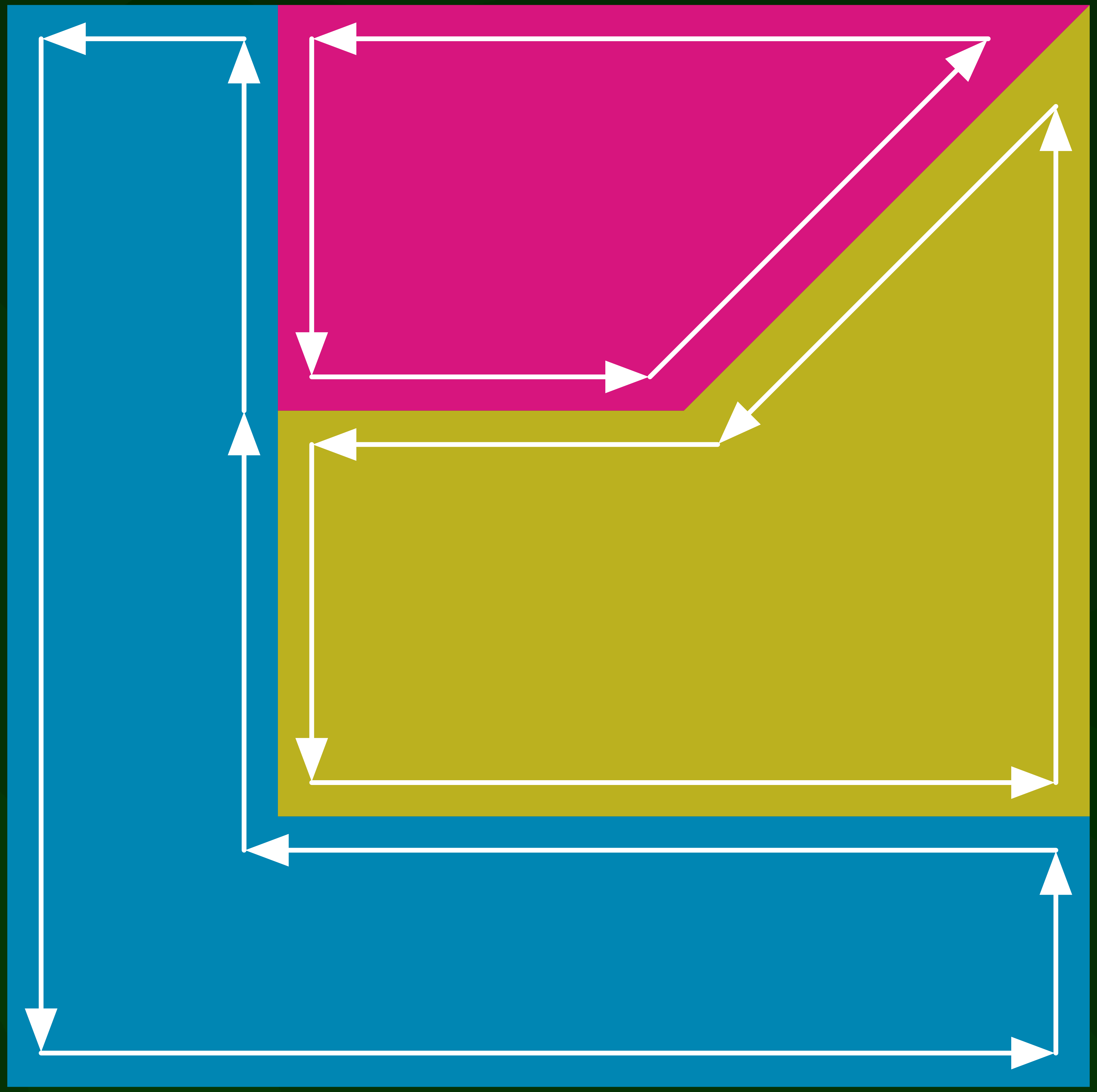
- Informally:
  - a generalised map dart is a unique combination of a cell of every dimension: vertex, edge, face, volume, …
  - a combinatorial map dart is a unique combination of a cell of every dimension from one upwards: edge, face, volume, …

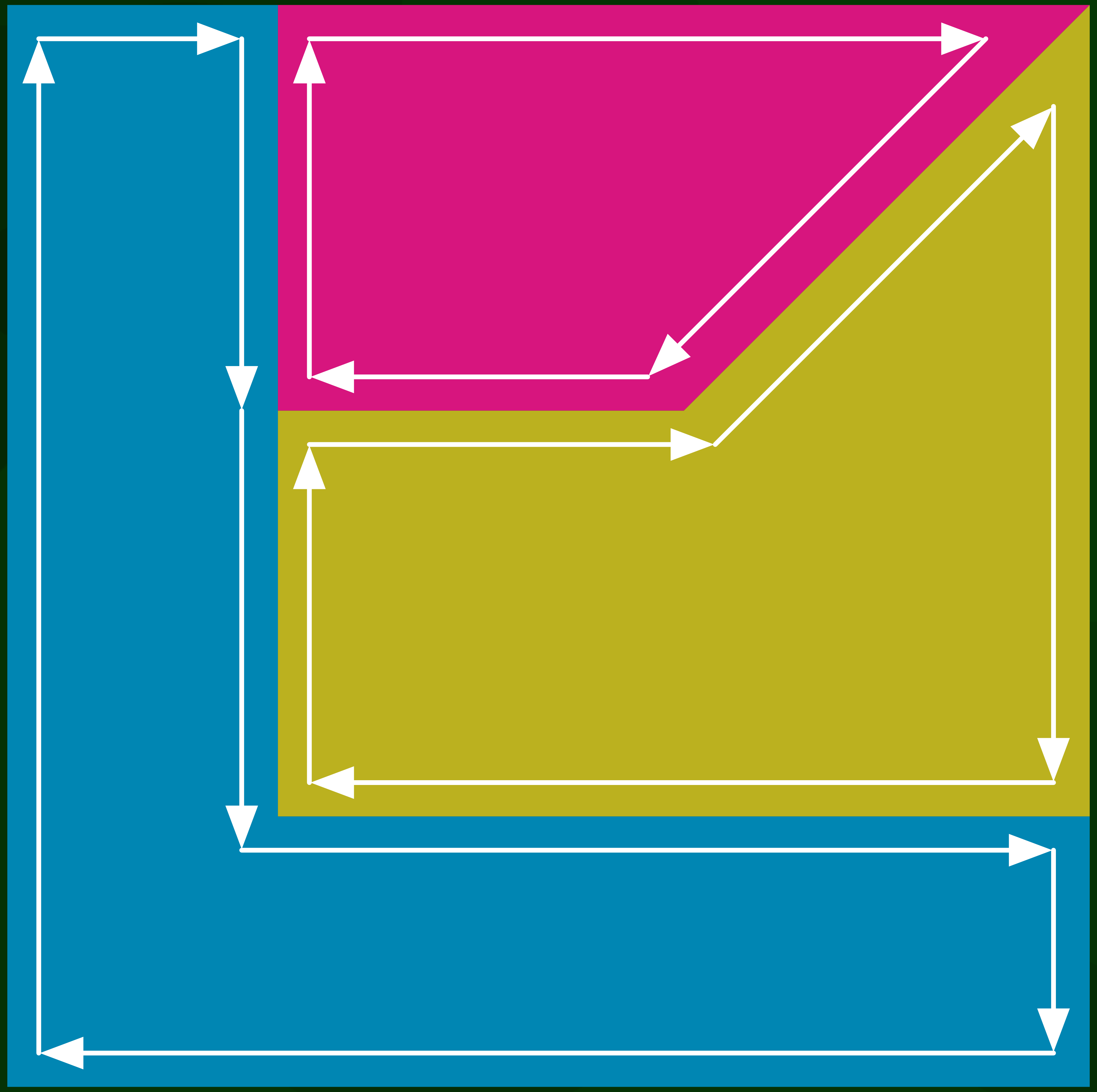
Intuitive meaning of a dart

- Why informal? only given certain conditions, e.g.
    - linear embeddings (i.e. polygons, not curved surfaces)
    - no bridge edges

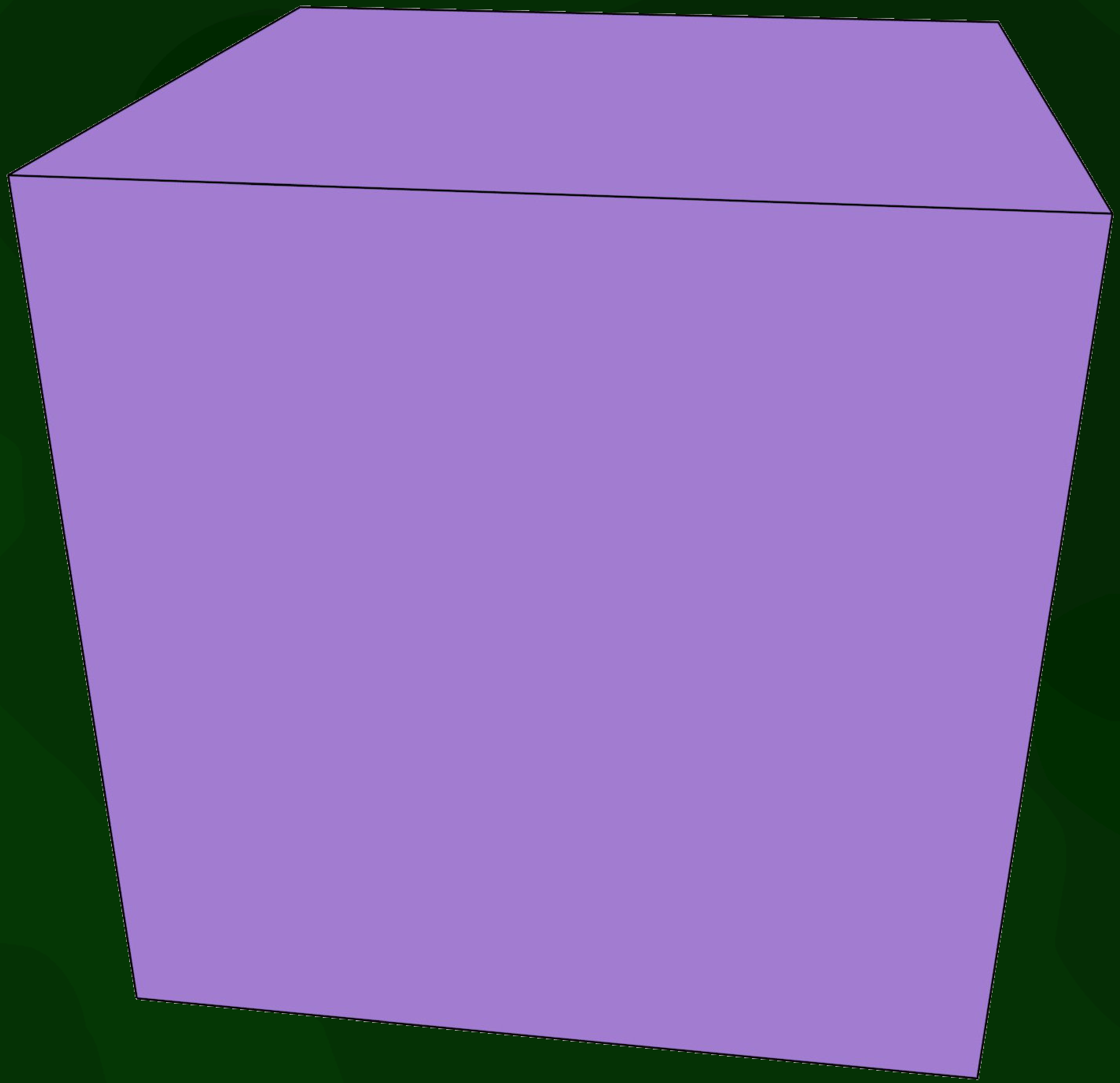


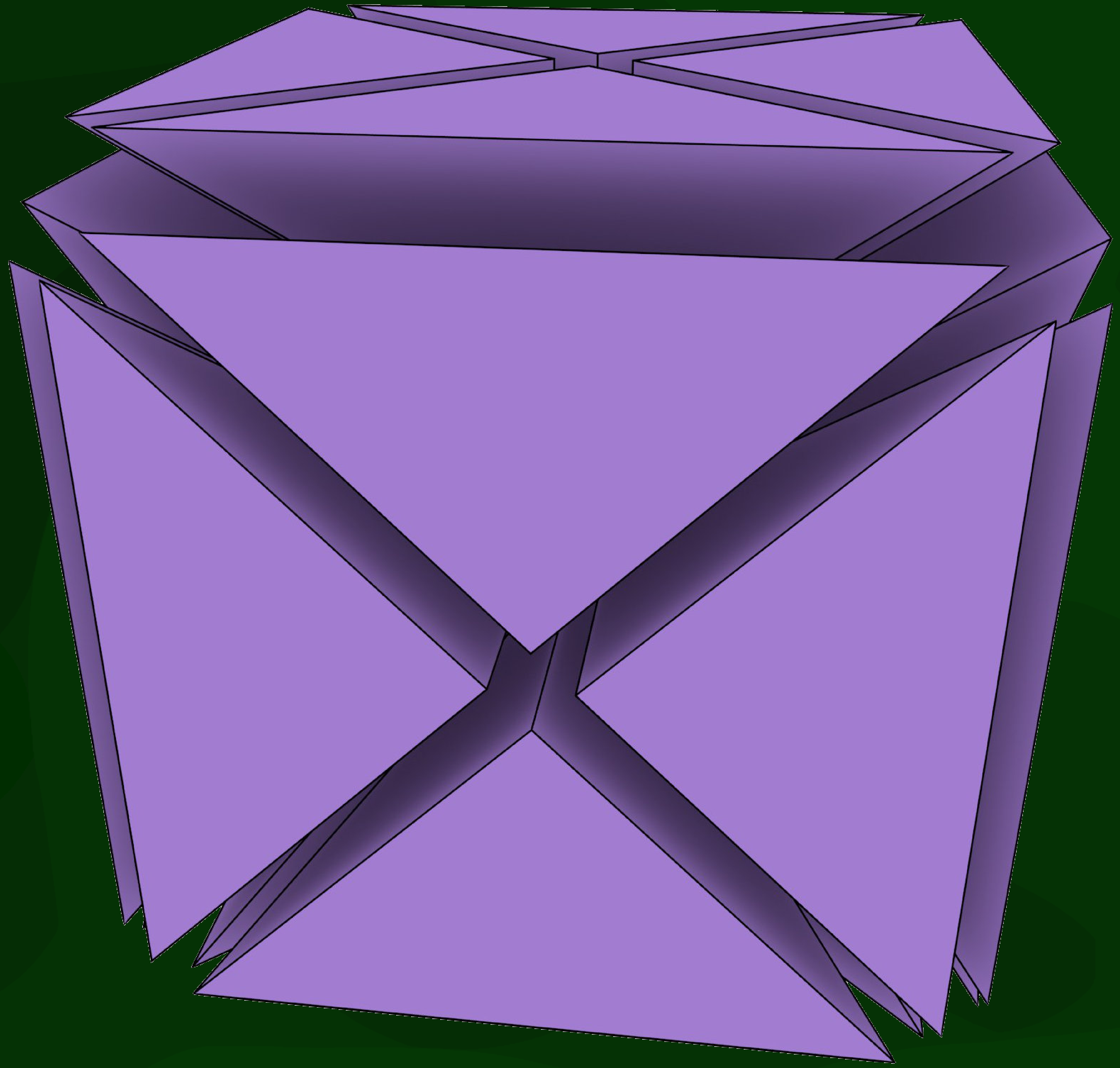
C-maps: orientation



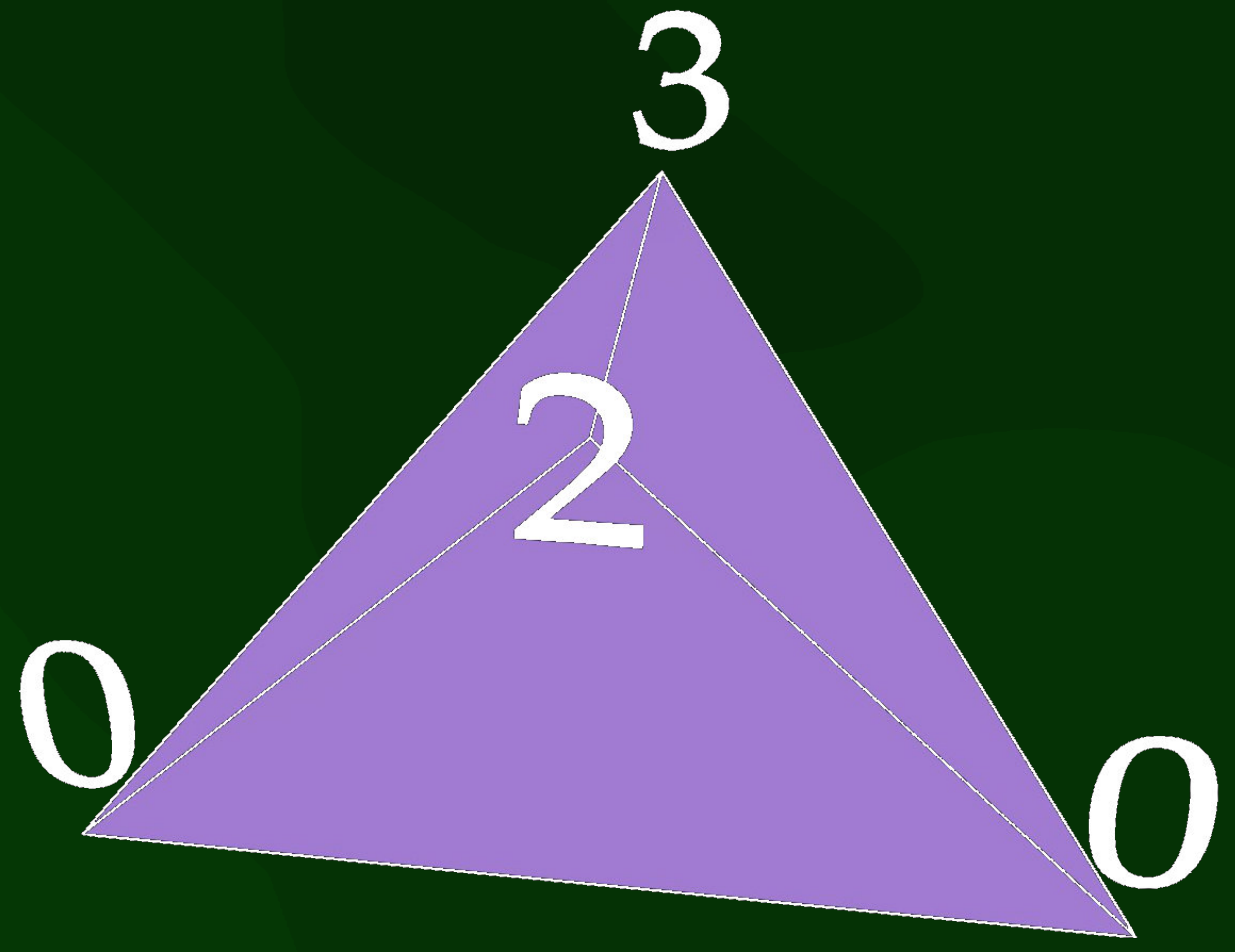


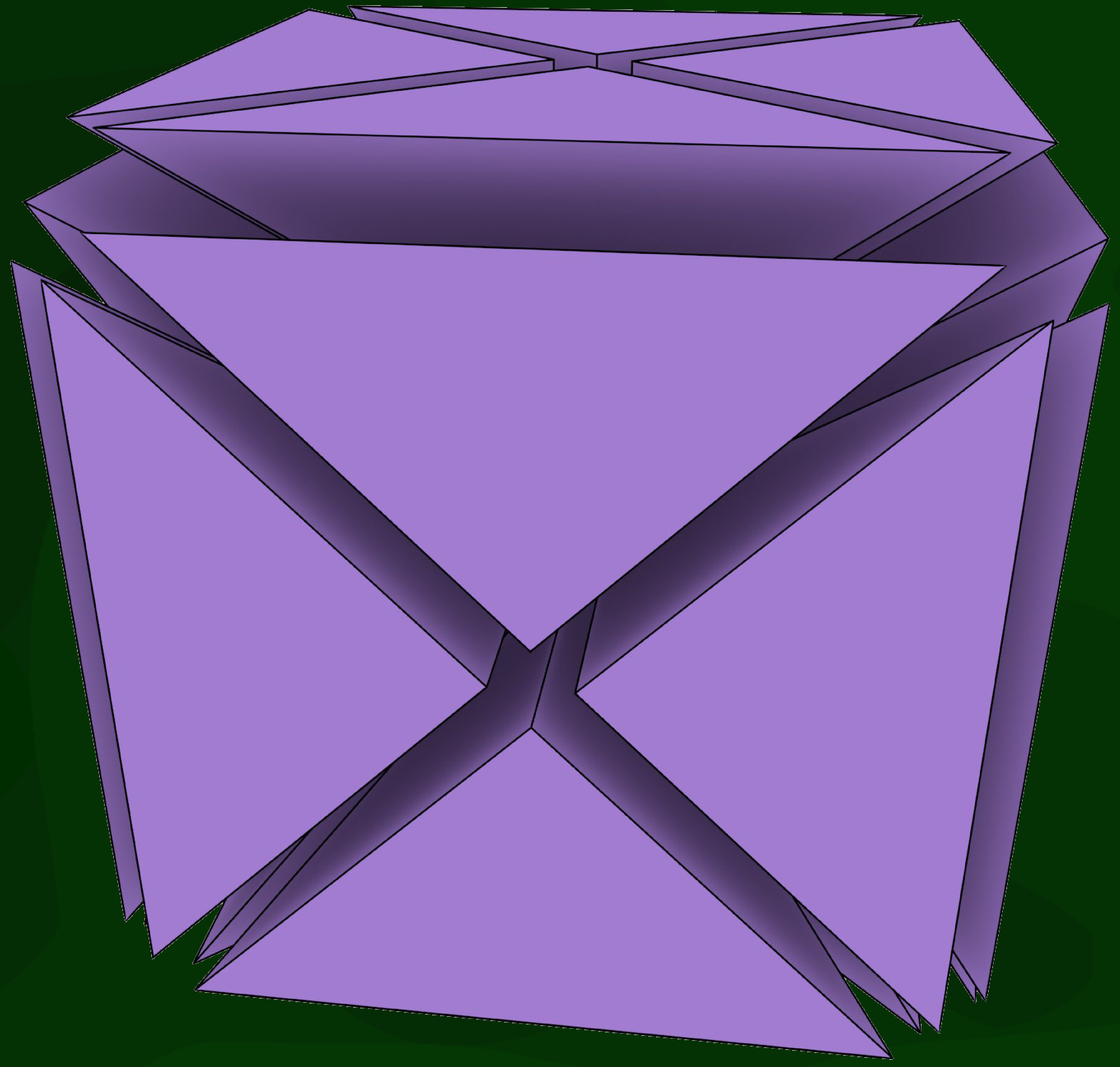
Building g-maps / c-maps





Building g-maps / c-maps





Traversing darts

- From the properties of simplices: a dart d in an nD combinatorial/generalised map has n+1 adjacent darts, each of which shares all but one of the vertices of d.

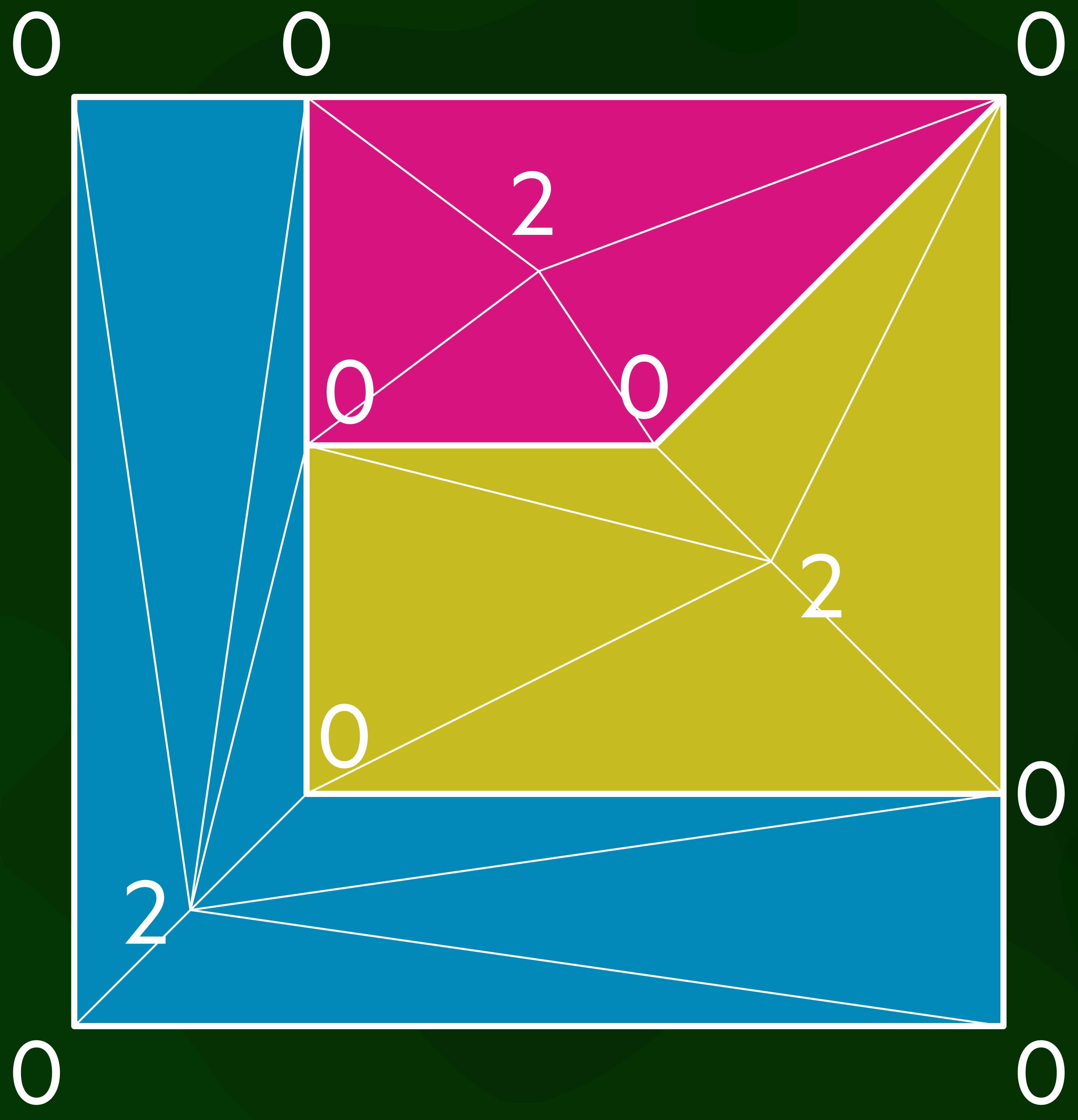
➔ Informally: Two adjacent darts share all of their cells except for one. e.g. if they differ in their edge, they share their vertex, face, volume, etc.

Traversing darts

- The link from an i-dimensional vertex of a dart d to the (other) i-dimensional vertex of its adjacent neighbour is called:
    - α_i in a generalised map
    - β_i in a combinatorial map
- Informally, it means switching the i-cell of d for the i-cell of its neighbour

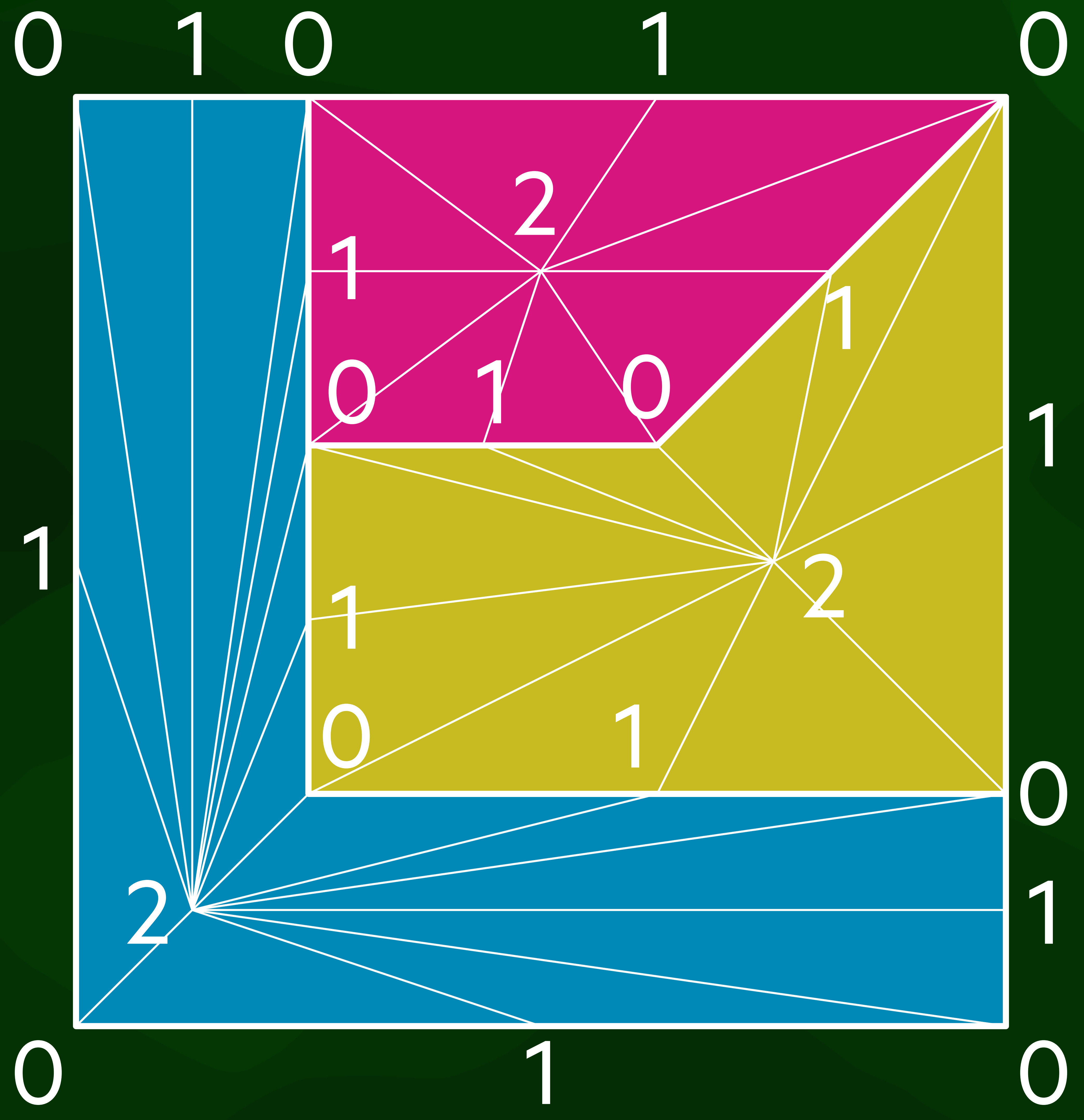
Traversing darts

- for all i, α_i is an involution
- for β > 1, β_i is also an involution
- but for for β = 1, β_i is a permutation



combinatorial map

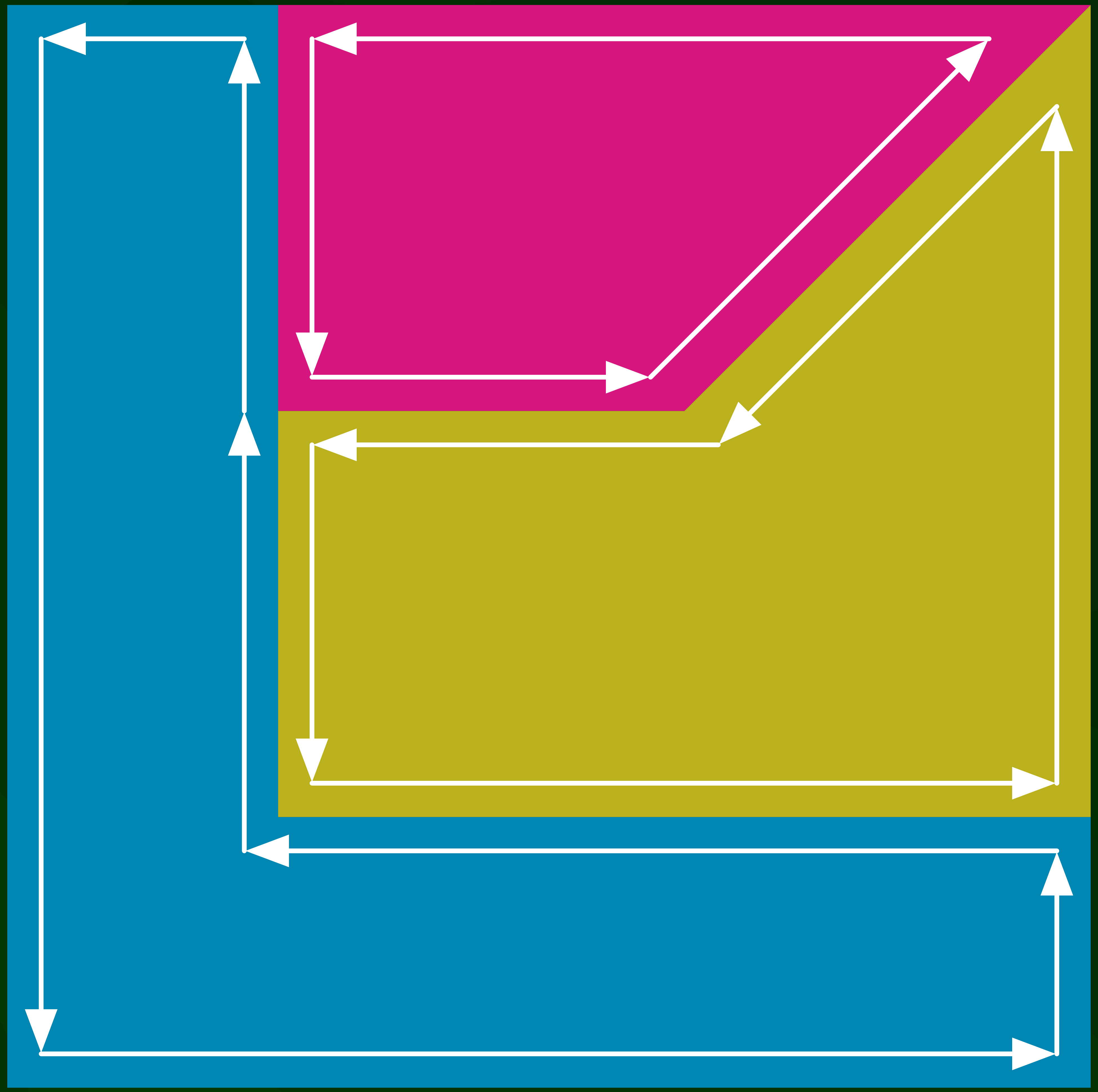
(c-map)



generalised map

(g-map)

C-maps vs. half-edge



- β_1 = next
- β_1^{-1} = prev
- β_2 = twin

Storage

- In a generalised map, a list of darts of the form:
  - d = [[$\alpha_0(d)$, $\alpha_1(d)$, $\alpha_2(d)$, ...],

    [a_0, a_1, a_2, ...]]
  - where each α is a link (ID, pointer) to another dart, and

- In a combinatorial map, a list of darts of the form:
  - d = [[β_1^{-1}, β_1, β_2, ...],

    [a_0, a_1, a_2, ...]]
  - where each β is a link (ID, pointer) to another dart, and

- each a is an optional link to a data structure with the attributes for the i-cell of d, including the coordinates in a_0.

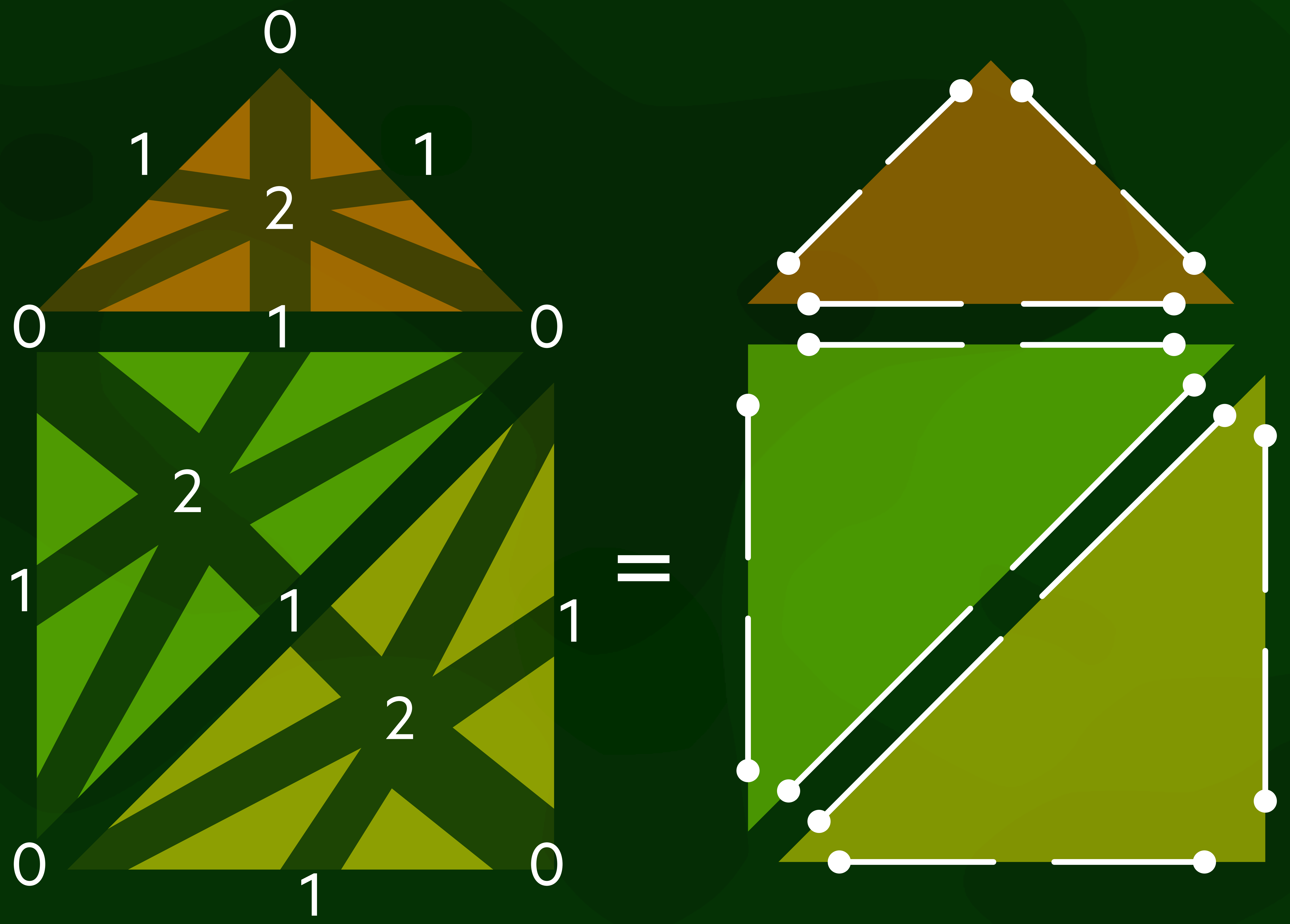
In practice? CGAL

https://doc.cgal.org/latest/Combinatorial_map/index.html#Chapter_Combinatorial_Maps

https://doc.cgal.org/latest/Generalized_map/index.html#Chapter_Generalized_Maps

https://doc.cgal.org/latest/Linear_cell_complex/index.html#Chapter_Linear_Cell_Complex

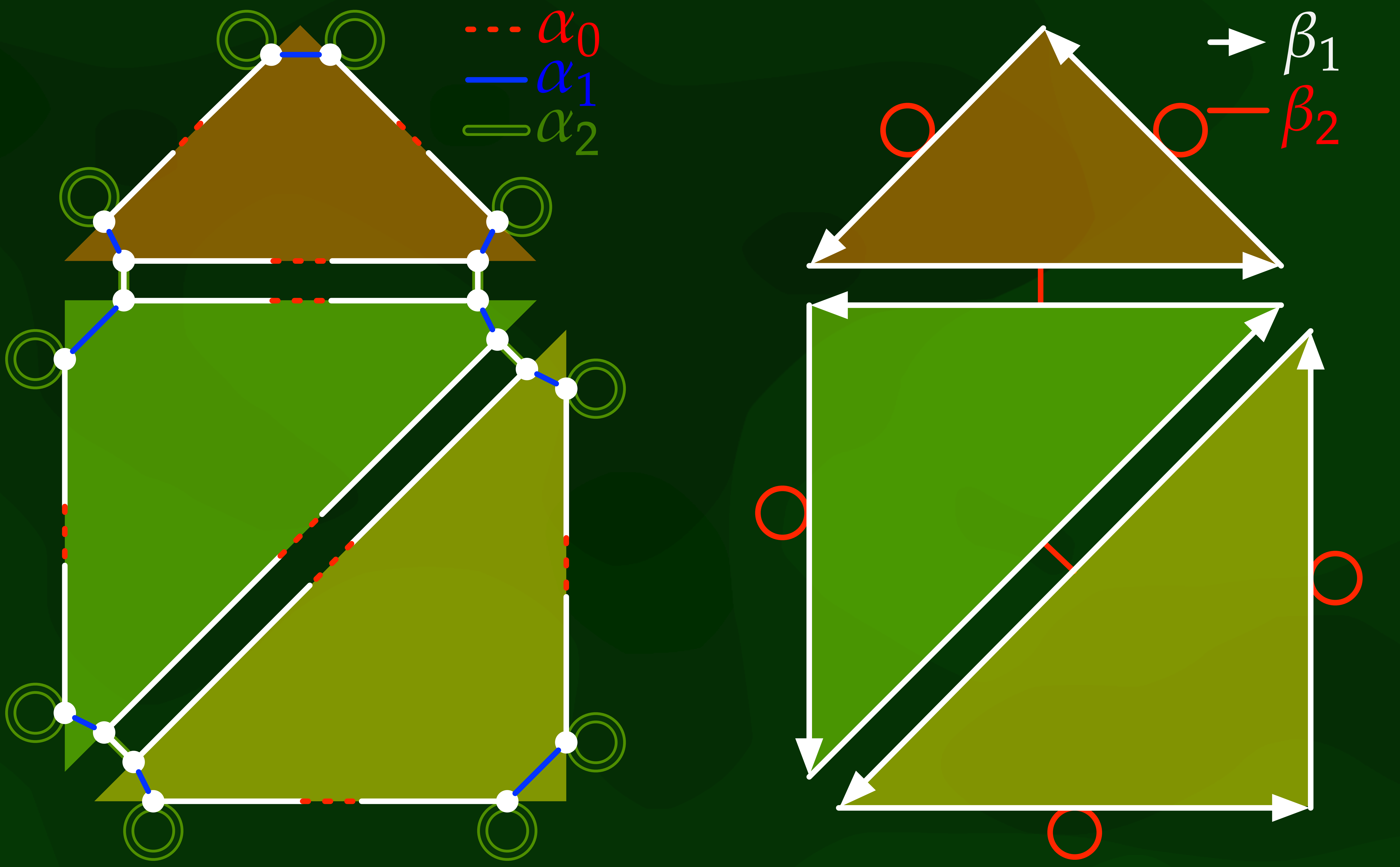
Simpler representation



Dart as simplex

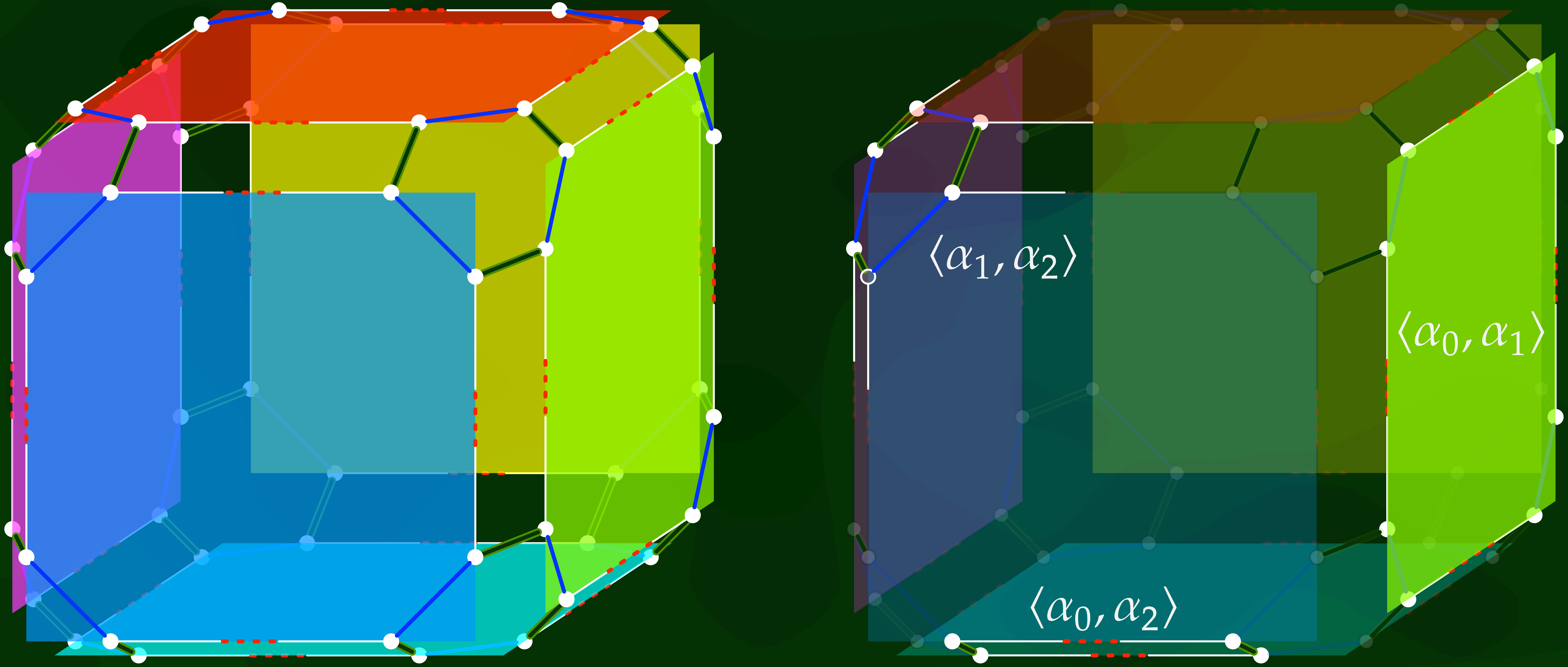
Dart as vertex-edge-face-...

Involutions and permutations



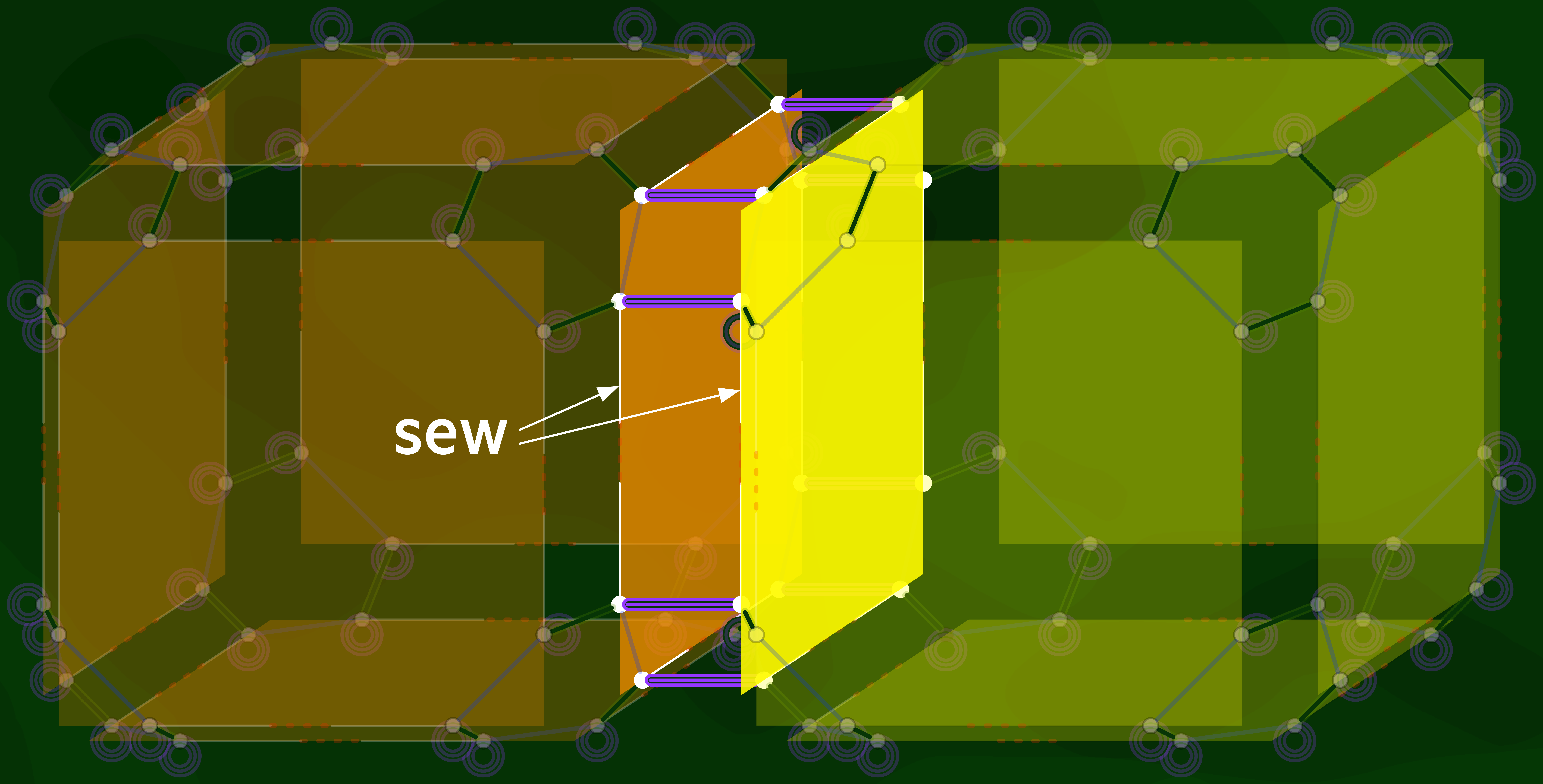
How to build more complex cmaps / gmaps?

Orbits



$\langle \alpha_1, \alpha_2 \rangle$
$\langle \alpha_0, \alpha_1 \rangle$
$\langle \alpha_0, \alpha_2 \rangle$

Sewing (3D)



Variations

- More complex embeddings for non-linear geometries, e.g. storing control points in edges or surfaces
- Storing non-manifolds with chains of maps

What to do next?

1. Today:
    - Go to geo1004 website and study today's lessons (updated 3D book Chapters 7 & 8)
    - Review materials for midterm exam (3D book Chapters 1-4 & 7-9)
2. Wednesday: CityJSON demos, intro to Hw 2

https://3d.bk.tudelft.nl/courses/geo1004

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Department of Urbanism
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