Linear Algebra - Part 3

1. Eigenvalues and eigenvectors
2. Singular value decomposition

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Eigenvalues and Eigenvectors

- What is the difference between the results of these multiplications?

$$\begin{bmatrix} 6 & 3 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 33 \\ 27 \end{bmatrix} \quad \text{vs.} \quad \begin{bmatrix} 6 & 3 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 30 \\ 40 \end{bmatrix} = 10 \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Eigenvalues and Eigenvectors

- Definition
  - An **eigenvector** of an $\boldsymbol{n} \times \boldsymbol{n}$ matrix A is a nonzero vector $\boldsymbol{v}$ such that $A\boldsymbol{v} = \lambda\boldsymbol{v}$ for some scalar λ. The scalar λ is called an eigenvalue of A if there is a nontrivial solution $\boldsymbol{v}$ of $A\boldsymbol{v} = \lambda\boldsymbol{v}$; such an $\boldsymbol{v}$ is called an eigenvector corresponding to λ.

$$A\boldsymbol{v} = \lambda\boldsymbol{v}$$

Eigenvalues and Eigenvectors

- Geometric meaning
  - Non-zero
  - When A applied to it, does not change direction
  - Only scaled by the scalar value λ

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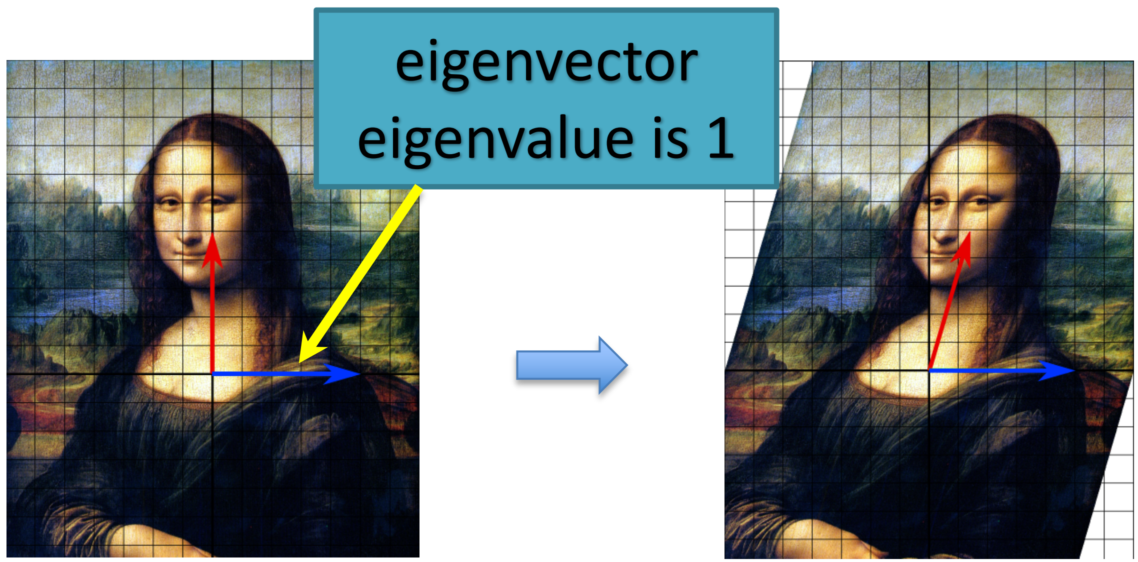
$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$



Shear transformation

Eigenvalues and Eigenvectors

$$Av = \lambda v$$ ⇨

Eigenvalues and Eigenvectors

$Av = \lambda v$ ⇨ $(A - \lambda I)v = 0$

I is the *n* by *n identity matrix*

v is non-zero ⇨ $\det(A - \lambda I) = 0$

- The eigenvalues of A are the roots of the characteristic equation

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Example: $M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Eigenvalues and Eigenvectors

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$$\det(A - \lambda I) = 0$$

Example: $M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. $|M - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 3 - 4\lambda + \lambda^2$.

Roots of $\lambda^2 - 4\lambda + 3 = 0$ are: $\lambda_1 = 1$ and $\lambda_2 = 3$

Eigenvalues and Eigenvectors

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Roots of $\lambda^2 - 4\lambda + 3 = 0$ are: $\lambda_1 = 1$ and $\lambda_2 = 3$

Eigenvector corresponding to $\lambda_1 = 1$ can obtained by solving $M\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow \begin{cases} 2x_1 + x_2 = x_1 \\ x_1 + 2x_2 = x_2 \end{cases} \rightarrow \begin{cases} x_1 + x_2 = 0 \\ x_1 + x_2 = 0 \end{cases} \rightarrow v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \ldots v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Eigenvalues and Eigenvectors

TU Delft 3Dgeoinfo

- Properties/Theorems
  - The trace of A, defined as the sum of its diagonal elements, is also the sum of all eigenvalues

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii} = \sum_{i=1}^{n} \lambda_i = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

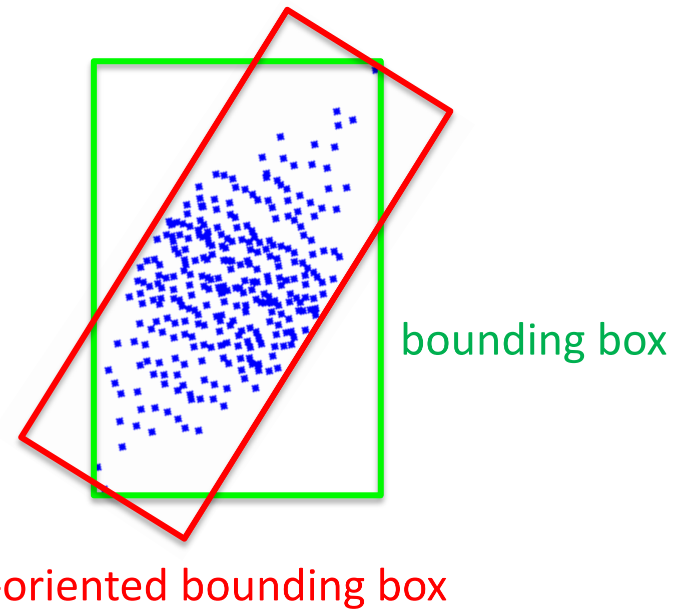
  - The determinant of A is the product of all its eigenvalues

$$\det(A) = \prod_{i=1}^{n} \lambda_i = \lambda_1 \lambda_2 \cdots \lambda_n$$

  - A invertible $\leftarrow\rightarrow$ every eigenvalue is nonzero
  - If A invertible, the eigenvalues of A^{-1} are $1/\lambda_1, 1/\lambda_2, \ldots, 1/\lambda_n$
  - A^T has the same eigenvalues A.

Eigenvalues and Eigenvectors

- Applications
  - Minimum enclosing rectangle (or **object-oriented bounding box**)



Eigenvalues and Eigenvectors

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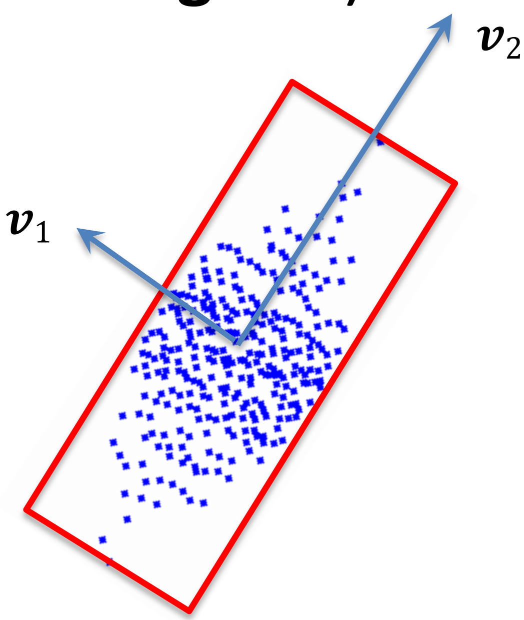


1) Construct covariance matrix

$$\boldsymbol{C} = \frac{1}{k}\sum_{i=1}^{k}(\boldsymbol{p}_i - \overline{\boldsymbol{p}}) \otimes (\boldsymbol{p}_i - \overline{\boldsymbol{p}})$$

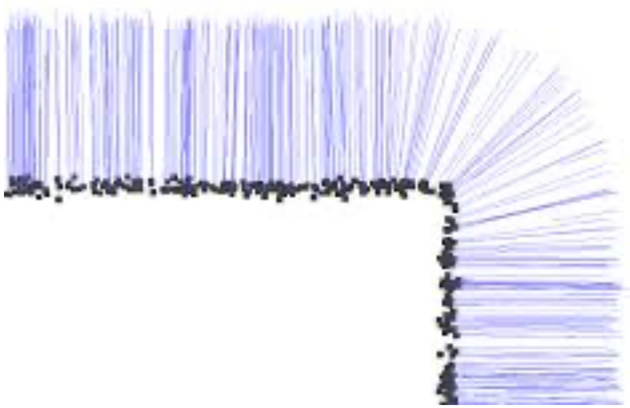
2) Compute eigenvalues and eigenvectors

$$\boldsymbol{C} \cdot \boldsymbol{v}_j = \boldsymbol{\lambda}_j \cdot \boldsymbol{v}_j,\ \ \boldsymbol{j} \in \{\boldsymbol{0}, \boldsymbol{1}, \boldsymbol{2}\}$$

3) The two directions of the eigenvectors give the axes
4) Project the points onto the two axes to determine their range

Eigenvalues and Eigenvectors

- Applications
  - Minimum enclosing rectangle (or **oriented bounding box**)
  - Normal estimation for point clouds



1) For each point, find it K closest neighbors
2) Construct covariance matrix

$$\boldsymbol{C} = \frac{\mathbf{1}}{\boldsymbol{k}} \sum_{i=1}^{k} (\boldsymbol{p}_i - \overline{\boldsymbol{p}}) \otimes (\boldsymbol{p}_i - \overline{\boldsymbol{p}})$$

3) Compute eigenvalues and eigenvectors

$$\boldsymbol{C} \cdot \boldsymbol{v}_j = \lambda_j \cdot \boldsymbol{v}_j,\ \ j \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$$

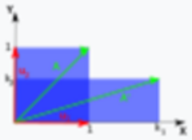
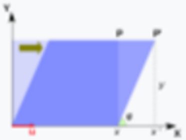
4) Normal vector = the eigenvector corresponding to the **smallest** eigenvalue

Eigenvalues and Eigenvectors

- Applications
  - Minimum enclosing rectangle (or **oriented bounding box**)
  - Normal estimation for point clouds
  - Inverse of matrix A
  - solve linear systems
  - matrix approximation
  - image compression
  - …

Eigenvalues and Eigenvectors

- Compute the eigenvalues and eigenvectors of the following transformations

	Scaling	Unequal scaling	Horizontal shear
Illustration			
Matrix	$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$	$\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

Eigenvalues and Eigenvectors

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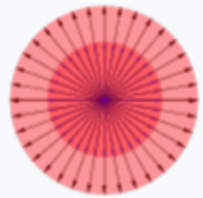
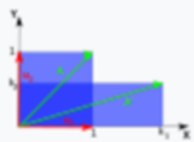
Illustration			
Matrix	$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$	$\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Eigenvalues, λ_i	$\lambda_1 = \lambda_2 = k$	$\lambda_1 = k_1$ $\lambda_2 = k_2$	$\lambda_1 = \lambda_2 = 1$
Eigenvectors	All non-zero vectors	$u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Matrix Decomposition

- LU decomposition

$$\mathbf{A} = \mathbf{LU} = \begin{bmatrix} 1 & 0 & 0 \\ L10 & 1 & 0 \\ L20 & L21 & 1 \end{bmatrix} \begin{bmatrix} U00 & U01 & U02 \\ 0 & U11 & U12 \\ 0 & 0 & U22 \end{bmatrix}$$

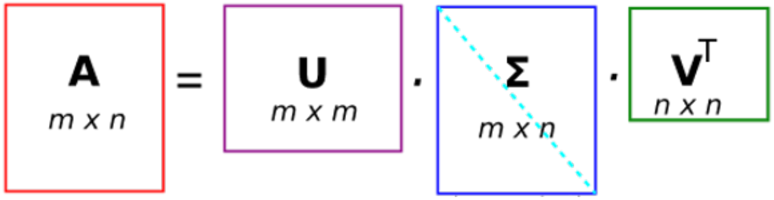
- Cholesky decomposition

$$\mathbf{A} = \mathbf{LL}^T = \begin{pmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \begin{pmatrix} L_{11} & L_{21} & L_{31} \\ 0 & L_{22} & L_{32} \\ 0 & 0 & L_{33} \end{pmatrix}$$

- Singular value decomposition

Singular Value Decomposition

- Definition
  - A *factorization* of a given matrix to its components



where

- A – an *m x n* real (or complex) matrix
- U – an *m x m* orthogonal *matrix*
- V – an *n x n* orthogonal *matrix*
- Σ – an *m x n* diagonal matrix; the entries on the diagonal called the **singular values**

Singular Value Decomposition

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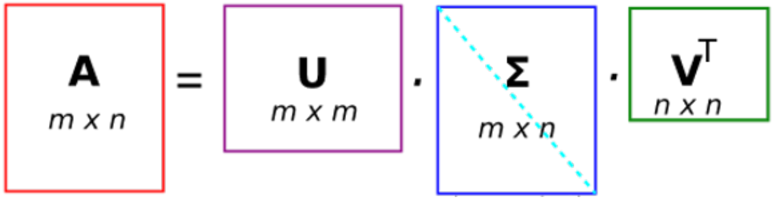

Example (general case)

$$\begin{bmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A U Σ V^T

Singular Value Decomposition

- Geometric meaning

Example (square matrix)

$$\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} -.40 & .916 \\ .916 & .40 \end{bmatrix} \cdot \begin{bmatrix} 5.39 & 0 \\ 0 & 3.154 \end{bmatrix} \cdot \begin{bmatrix} -.05 & .999 \\ .999 & .05 \end{bmatrix}$$

A	U	Σ	V^T
Transformation	Rotation	Scaling	Rotation

Singular Value Decomposition

- Geometric meaning

$$A = U \Sigma V^T$$

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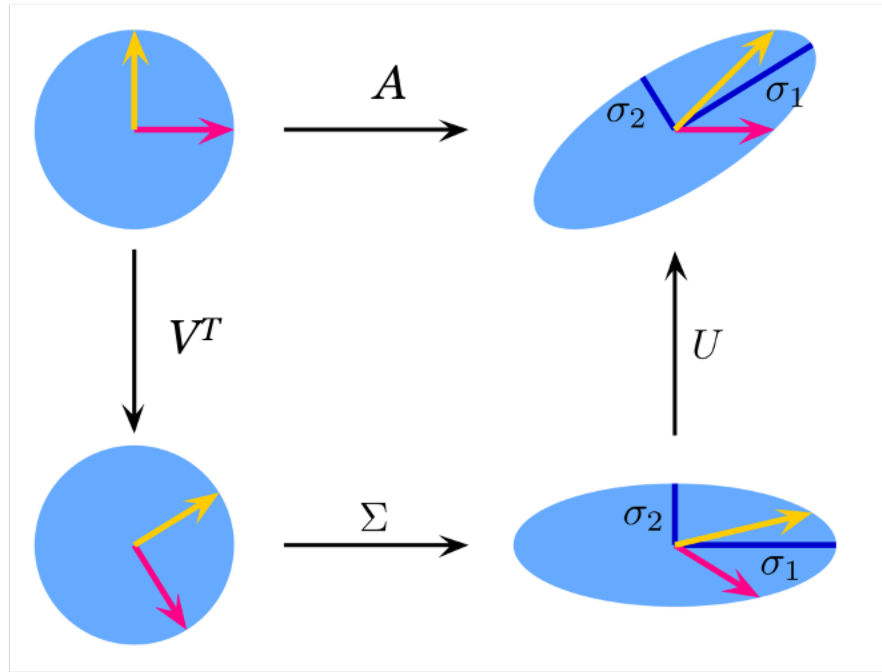

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Transformation	Rotation	Scaling	Rotation

Singular Value Decomposition

- How to compute?
  - Theorem: SVD and eigenvalues/eigenvectors
    - The columns of **U** are eigenvectors of $\mathbf{AA}^T$
    - The columns of **V** are eigenvectors of $\mathbf{A}^T\mathbf{A}$
    - The non-zero elements of Σ are the square roots of the non-zero eigenvalues of $\mathbf{A}^T\mathbf{A}$ or $\mathbf{AA}^T$

$$A = U \Sigma V^T \qquad \Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \cdot & \\ & & & \sigma_N \end{bmatrix}$$

U, V = orthogonal matrix

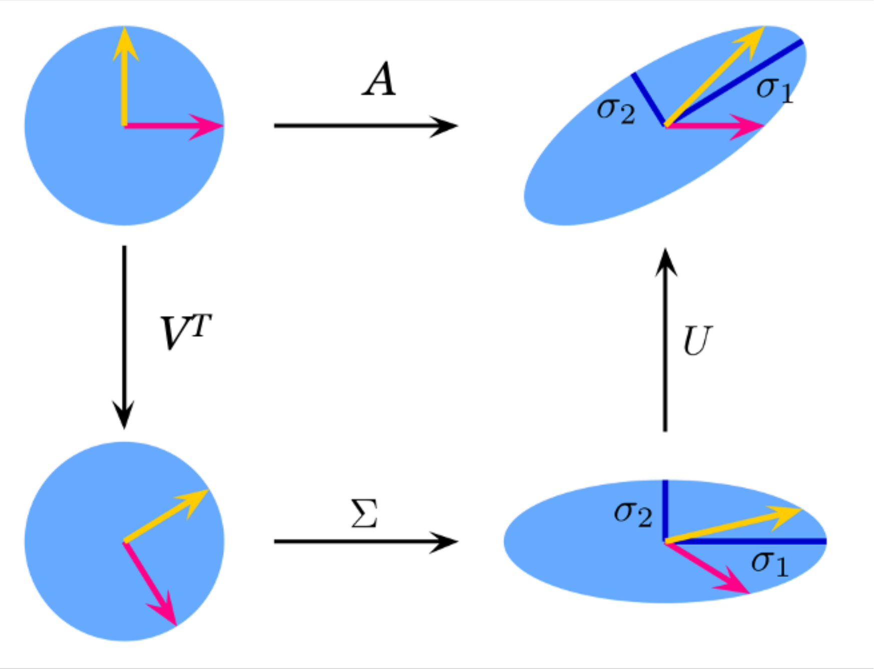
$$\sigma_i = \sqrt{\lambda_i}$$

σ = singular value
λ = eigenvalue of $A^t A$

Singular Value Decomposition

- Applications
  - Transformation decomposition

$A = U \Sigma V^T$



Singular Value Decomposition

- Applications
  - Transformation decomposition
  - Solve homogenous linear systems $A\boldsymbol{x} = 0$
    - A is a square matrix
    - $\boldsymbol{x}$ = 0 is always a valid solution
    - det (A) = 0 → a non-zero solution
      - $A = U \Sigma V^T$
      - $\boldsymbol{x}$: the last column of ***V*** (i.e., right *singular vector* corresponding to the zero *singular value of A*)

Singular Value Decomposition

- Applications
  - Transformation decomposition
  - Solve homogenous linear systems $A\boldsymbol{x} = 0$
  - Compute inverse of a matrix A

$$A = U \Sigma V^T$$

Singular Value Decomposition

- Applications
  - Transformation decomposition
  - Solve homogenous linear systems $Ax = 0$
  - Compute inverse of a matrix A

$$A = U \Sigma V^T$$

$$\rightarrow A^{-1} = (U \Sigma V^T)^{-1}$$

$$= (V^T)^{-1} \Sigma^{-1} U^{-1}$$

$$= V \Sigma^{-1} U^T$$

U and V orthogonal, so inverse = transpose

Σ^{-1} is also diagonal with reciprocals of entries of Σ

Singular Value Decomposition

- Applications
  - Transformation decomposition
  - Solve homogenous linear systems $A\boldsymbol{x} = 0$
  - Compute inverse of a matrix A
  - Camera calibration (in GEO1016)
    - Recover the camera parameters from a set of 3D-pixel correspondences.

Assignment 3: part 3

Question 7: eigenvalues, eigenvectors

Question 8: transformation decomposition

Hint: do some (simply) transformation of SVD

Exam (linear algebra part)

1. 2 or 3 multi-choice questions
   - With four choices A, B, C, and D
   - Only one correct answer
2. 2 or 3 open-ended questions
   - Give an answer
   - Give the explanation