Linear Algebra - Part 2
Linear systems, linear least-squares

Liangliang Nan

Linear system

- Linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

$x_1, \ldots, x_n$: variables

b and $a_1, \ldots, a_n$: constant real (or complex) numbers

- Linear system (a system of linear equations)

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots &= b_3 \\ &\vdots \end{aligned}$$

Linear system

- Solution of a linear system
  - A list of numbers satisfying each equation

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots &= b_3 \\
&\vdots
\end{aligned}
$$

Linear system

- Solution of a linear system
  - A list of numbers satisfying each equation

Example

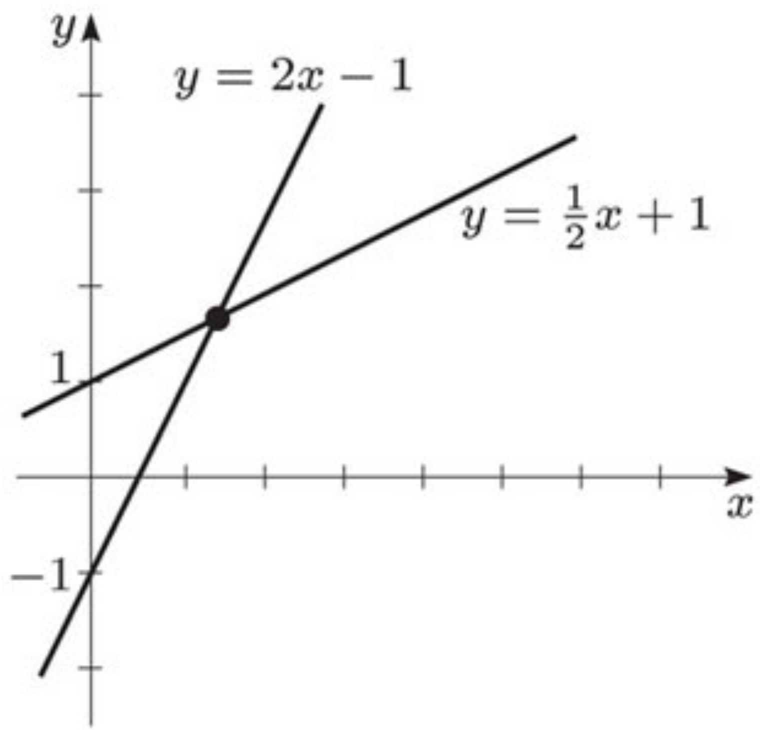
$$\begin{aligned} 2x - y &= 1 \\ x - 2y &= -2 \end{aligned}$$

Linear system

- Solution of a linear system
  - A list of numbers satisfying each equation

Example (unique solution)

$$2x - y = 1$$
$$x - 2y = -2$$



Linear system

- Solution of a linear system
  - A list of numbers satisfying each equation
- Solution set
  - The set of all possible solutions

Example

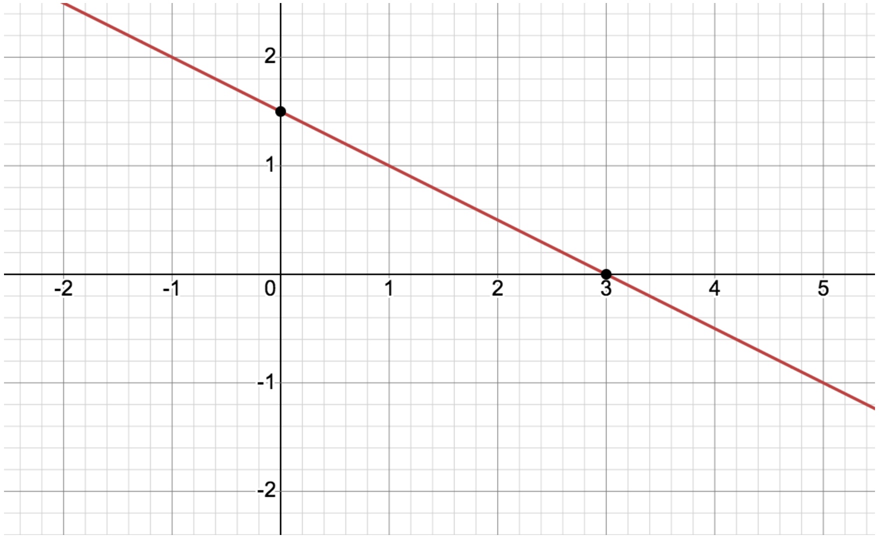
$$x + 2y = 3$$

Linear system

- Solution of a linear system
  - A list of numbers satisfying each equation
- Solution set
  - The set of all possible solutions

Example (infinite number of solutions)

$$x + 2y = 3$$



Linear system

- Solution of a linear system
  - A list of numbers satisfying each equation
- Solution set
  - The set of all possible solutions

Example

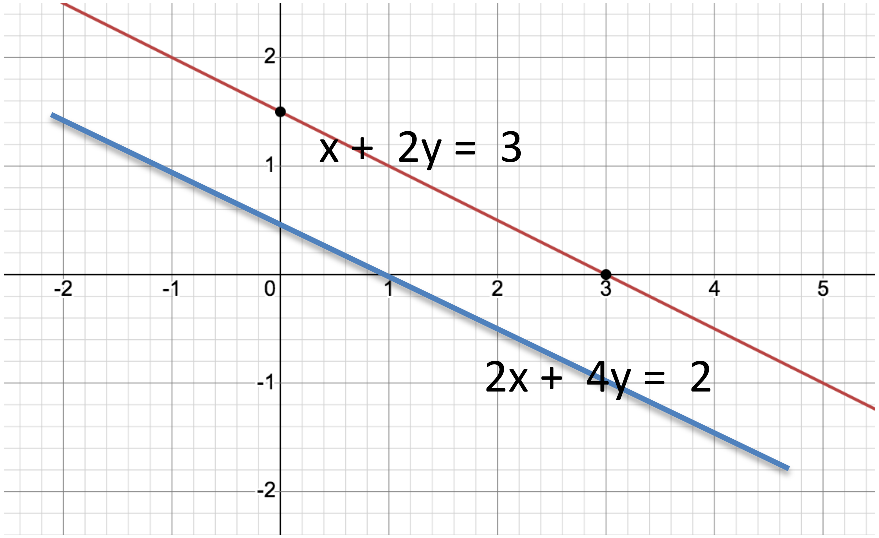
$$\begin{aligned} x + 2y &= 3 \\ 2x + 4y &= 2 \end{aligned}$$

Linear system

- Solution of a linear system
  - A list of numbers satisfying each equation
- Solution set
  - The set of all possible solutions

Example (no solution)

$$\begin{aligned} x + 2y &= 3 \\ 2x + 4y &= 2 \end{aligned}$$

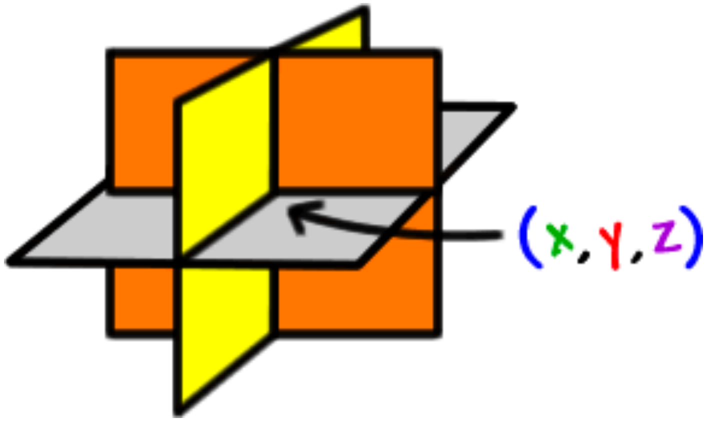


Linear system

- A system of linear equations has
  - no solution, or
  - exactly one solution, or
  - infinitely many solutions.
- A system of linear equations is **consistent**
  - One solution or infinitely many solutions
- A system of linear equations is **inconsistent**
  - No solution

Linear system

- Does a system of linear equations have
  - no solution, or
  - exactly one solution, or
  - infinitely many solutions ?
- If unique solution, what is it?



Solve a linear system

- Without/With matrix notation

Solve a linear system

- Without matrix notation
  - Row transformation: elimination

$$x_1 - 2x_2 + x_3 = 0 \quad \text{----(1)}$$

$$2x_2 - 8x_3 = 8 \quad \text{----(2)}$$

$$-4x_1 + 5x_2 + 9x_3 = -9, \quad \text{----(3)}$$

Solve a linear system

- Without matrix notation
  - Row transformation: elimination

$$x_1 - 2x_2 + x_3 = 0 \quad \text{----(1)}$$
$$2x_2 - 8x_3 = 8 \quad \text{----(2)}$$
$$-4x_1 + 5x_2 + 9x_3 = -9, \quad \text{----(3)}$$

→

$$x_1 - 2x_2 + x_3 = 0 \quad \text{----(1)}$$
$$2x_2 - 8x_3 = 8 \quad \text{----(2)}$$
$$-3x_2 + 13x_3 = -9 \quad \text{----(3)}$$

(1) * 4 + (3)

Solve a linear system

- Without matrix notation
  - Row transformation: elimination

$$x_1 - 2x_2 + x_3 = 0 \quad \text{----(1)}$$
$$2x_2 - 8x_3 = 8 \quad \text{----(2)}$$
$$-4x_1 + 5x_2 + 9x_3 = -9, \quad \text{----(3)}$$

$\Rightarrow$

$$x_1 - 2x_2 + x_3 = 0 \quad \text{----(1)}$$
$$2x_2 - 8x_3 = 8 \quad \text{----(2)}$$
$$-3x_2 + 13x_3 = -9 \quad \text{----(3)}$$

(1) * 4 + (3)

$\Rightarrow$

$$x_1 - 2x_2 + x_3 = 0 \quad \text{----(1)}$$
$$2x_2 - 8x_3 = 8 \quad \text{----(2)}$$
$$x_3 = 3 \quad \text{----(3)}$$

(2) * 1.5 + (3)

Solve a linear system

- Without matrix notation
  - Row transformation: elimination

$$x_1 - 2x_2 + x_3 = 0 \quad \text{----(1)}$$
$$2x_2 - 8x_3 = 8 \quad \text{----(2)}$$
$$-4x_1 + 5x_2 + 9x_3 = -9, \quad \text{----(3)}$$

$\Rightarrow$

$$x_1 - 2x_2 + x_3 = 0 \quad \text{----(1)}$$
$$2x_2 - 8x_3 = 8 \quad \text{----(2)}$$
$$-3x_2 + 13x_3 = -9 \quad \text{----(3)}$$

(1) * 4 + (3)

$\Rightarrow$

$$x_1 - 2x_2 + x_3 = 0 \quad \text{----(1)}$$
$$2x_2 - 8x_3 = 8 \quad \text{----(2)}$$
$$x_3 = 3 \quad \text{----(3)}$$

(2) * 1.5 + (3)

$\Rightarrow$

$$x_1 - 2x_2 + x_3 = 0 \quad \text{----(1)}$$
$$x_2 = 16 \quad \text{----(2)}$$
$$x_3 = 3 \quad \text{----(3)}$$

(2) + 8 * (3)

$\Rightarrow$

$$x_1 = 29$$
$$x_2 = 16$$
$$x_3 = 3$$

Solve a linear system

- Matrix notation

Solve a linear system

- Matrix notation
  - Coefficient matrix

$$
\begin{aligned}
x_1 - 2x_2 + x_3 &= 0 \\
2x_2 - 8x_3 &= 8 \\
-4x_1 + 5x_2 + 9x_3 &= -9,
\end{aligned}
$$

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 8 \\ -9 \end{bmatrix}$$

$$A\,\mathbf{x} = \mathbf{b}$$

Solve a linear system

- Gaussian elimination
- Matrix inversion

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 8 \\ -9 \end{bmatrix}$$

$$A\,\mathbf{x} = \mathbf{b}$$

Solve a linear system

- Gaussian elimination
  - Strategy: find an **equivalent system** easier to solve

$$A\,\mathbf{x}=\mathbf{b} \;\Rightarrow\; A'\,\mathbf{x}=\mathbf{b}'$$

Solve a linear system

- Augmented matrix

A **x**=**b**, where $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$ $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\mathbf{b} = \begin{bmatrix} 0 \\ 8 \\ -9 \end{bmatrix}$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

[Ab]

Solve a linear system

- Gaussian elimination
  - Elementary row operations
    - **Replacement:** Replace one row by the sum of itself and a multiple of another row.
    - **Interchange:** Interchange two rows.
    - **Scaling:** Multiply all entries in a row by a nonzero constant.

Solve a linear system

- Gaussian elimination
  - Elementary row operations
    - **Replacement:** Replace one row by the sum of itself and a multiple of another row.
    - **Interchange:** Interchange two rows.
    - **Scaling:** Multiply all entries in a row by a nonzero constant.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \begin{matrix} \text{----(1)} \\ \text{----(2)} \\ \text{----(3)} \end{matrix}$$

Solve a linear system

- Gaussian elimination
  - Elementary row operations
    - **Replacement:** Replace one row by the sum of itself and a multiple of another row.
    - **Interchange:** Interchange two rows.
    - **Scaling:** Multiply all entries in a row by a nonzero constant.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \begin{matrix} \text{----(1)} \\ \text{----(2)} \\ \text{----(3)} \end{matrix} \quad \xrightarrow{(1)*4+(3) \to (3)} \quad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

Solve a linear system

- Gaussian elimination
  - Elementary row operations
    - **Replacement:** Replace one row by the sum of itself and a multiple of another row.
    - **Interchange:** Interchange two rows.
    - **Scaling:** Multiply all entries in a row by a nonzero constant.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \begin{matrix} \text{----(1)} \\ \text{----(2)} \\ \text{----(3)} \end{matrix} \xrightarrow{(1)*4+(3) \to (3)} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix} \xrightarrow{(2)*1/2 \to (2)} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

Solve a linear system

- Gaussian elimination
  - Elementary row operations
    - **Replacement:** Replace one row by the sum of itself and a multiple of another row.
    - **Interchange:** Interchange two rows.
    - **Scaling:** Multiply all entries in a row by a nonzero constant.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \begin{matrix} \text{----(1)} \\ \text{----(2)} \\ \text{----(3)} \end{matrix} \xrightarrow{(1)*4+(3) \to (3)} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix} \xrightarrow{(2)*1/2 \to (2)} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

$$\xrightarrow{(2)*3+(3) \to (3)} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Solve a linear system

- Gaussian elimination
  - Elementary row operations
    - **Replacement:** Replace one row by the sum of itself and a multiple of another row.
    - **Interchange:** Interchange two rows.
    - **Scaling:** Multiply all entries in a row by a nonzero constant.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \begin{matrix} \text{----(1)} \\ \text{----(2)} \\ \text{----(3)} \end{matrix}$$

(1)*4+(3) -> (3) ➡

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

(2)*1/2 -> (2) ➡

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

(2)*3+(3) -> (3) ➡

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

(1)-(3) -> (1)

(2)+4*(3) -> (2) ➡

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Solve a linear system

- Gaussian elimination
  - Elementary row operations
    - **Replacement:** Replace one row by the sum of itself and a multiple of another row.
    - **Interchange:** Interchange two rows.
    - **Scaling:** Multiply all entries in a row by a nonzero constant.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \begin{matrix} \text{----(1)} \\ \text{----(2)} \\ \text{----(3)} \end{matrix} \xrightarrow{(1)*4+(3) \to (3)} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix} \xrightarrow{(2)*1/2 \to (2)} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

$$\xrightarrow{(2)*3+(3) \to (3)} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow[(2)+4*(3) \to (2)]{(1)-(3) \to (1)} \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{(1)+2*(2) \to (1)} \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Solve a linear system

- Gaussian elimination

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\mathbf{I}\,\mathbf{x} = \begin{bmatrix} 29 \\ 16 \\ 3 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$x_1 = 29$$
$$x_2 = 16$$
$$x_3 = 3$$

Solve a linear system

- Gaussian elimination

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix} \qquad \mathbf{I}\,\mathbf{x} = \begin{bmatrix} 29 \\ 16 \\ 3 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \begin{aligned} x_1 &= 29 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9, \end{aligned} \qquad \begin{aligned} (29) - 2(16) + (3) = 29 - 32 + 3 &= 0 \\ 2(16) - 8(3) = 32 - 24 &= 8 \\ -4(29) + 5(16) + 9(3) = -116 + 80 + 27 &= -9 \end{aligned}$$

Solve a linear system

- Gaussian elimination

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

original augmented matrix → transformed augmented matrix

$$A\,\mathbf{x}=\mathbf{b} \Rightarrow \mathbf{I}\,\mathbf{x}=\mathbf{x}^0$$

original linear system → equivalent linear system

Solve a linear system

- Gaussian elimination
- Matrix inversion

Gaussian elimination

$A\,\mathbf{x}=\mathbf{b}$ → $\mathbf{I}\,\mathbf{x}=\mathbf{x}^0$

original linear system

equivalent linear system

Matrix inversion

$A^{-1}A\,\mathbf{x} = A^{-1}\mathbf{b}$

$\mathbf{x}= A^{-1}\mathbf{b}$

Direct solution

Solve a linear system

- Gaussian elimination
- Matrix inversion

Gaussian elimination

$A\,\mathbf{x}=\mathbf{b}$ → $\mathbf{I}\,\mathbf{x}=\mathbf{x}^0$

original linear system

equivalent linear system

Matrix inversion

$A^{-1}A\,\mathbf{x} = A^{-1}\mathbf{b}$

$\mathbf{x}= A^{-1}\mathbf{b}$

Direct solution

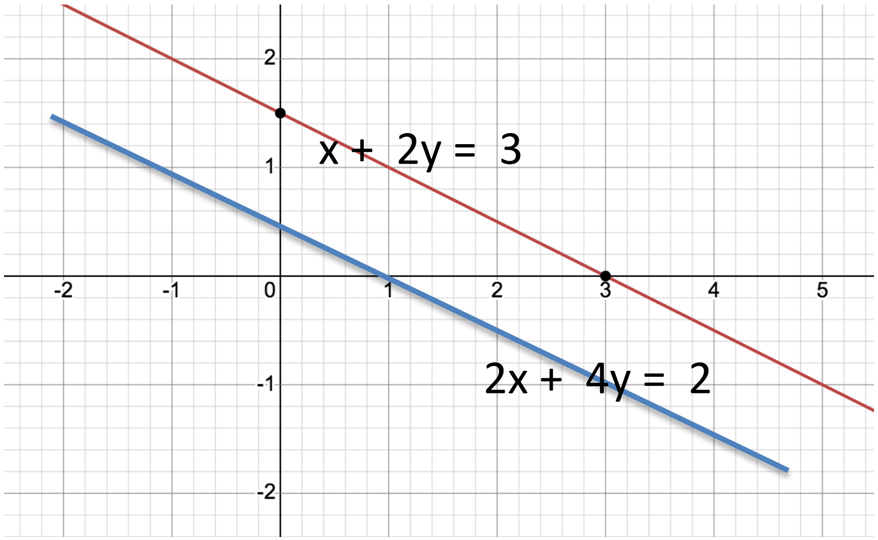
Useful only if unique solution exists

Gain insights before you solve

Linear system

- Two fundamental questions
  - Existence: Is the system consistent (does at least one solution *exist)?*
  - Uniqueness: If a solution exists, is it the *only* one?

$$\begin{aligned} x + 2y &= 3 \quad \text{----(1)} \\ 2x + 4y &= 2 \quad \text{----(2)} \end{aligned}$$



Linear system

- Two fundamental questions
  - Existence: Is the system consistent (does at least one solution *exist)?*
  - Uniqueness: If a solution exists, is it the *only* one?

- det(A) != 0 -> unique solution
- det(A) ==0 -> no solution, or many solution

Linear system

- Two fundamental questions
  - Existence: Is the system consistent (does at least one solution *exist)?*
  - Uniqueness: If a solution exists, is it the *only* one?

- det(A) != 0 -> unique solution
- det(A) ==0 -> no solution, or many solution

$$\begin{aligned} x + 2y &= 1 \\ 2x + 4y &= 2 \end{aligned}$$

Linear system

- Non-square systems

$$A_{m \times n}\,\mathbf{x}=\mathbf{b} \quad (\mathbf{m} \neq \mathbf{n})$$

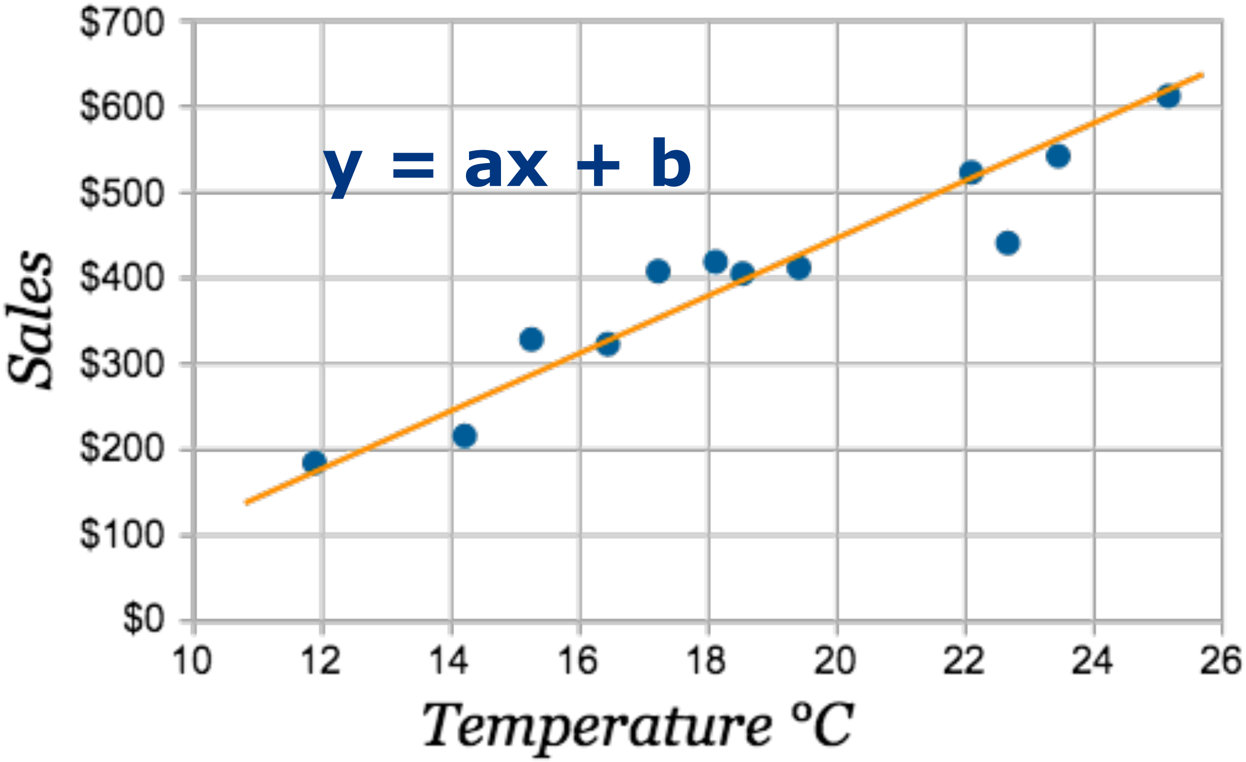
  - **Underdetermined** if m < n
    - "more unknowns than equations"
    - If any two equations are not consistent: no solution
    - If all pairs of equations are consistent: infinite number of solutions
  - **Overdetermined** if m>n
    - "more equations than unknowns"
    - Usually has no solution, but people may want the "best approximate solution"

Linear least squares

- Fit a mathematical model to data
  - Idealized value of the model for any data is linearly in term of the unknown parameters
  - The fitted model summarizes data, can predict unobserved values

Linear least squares

- Fit a mathematical model to data



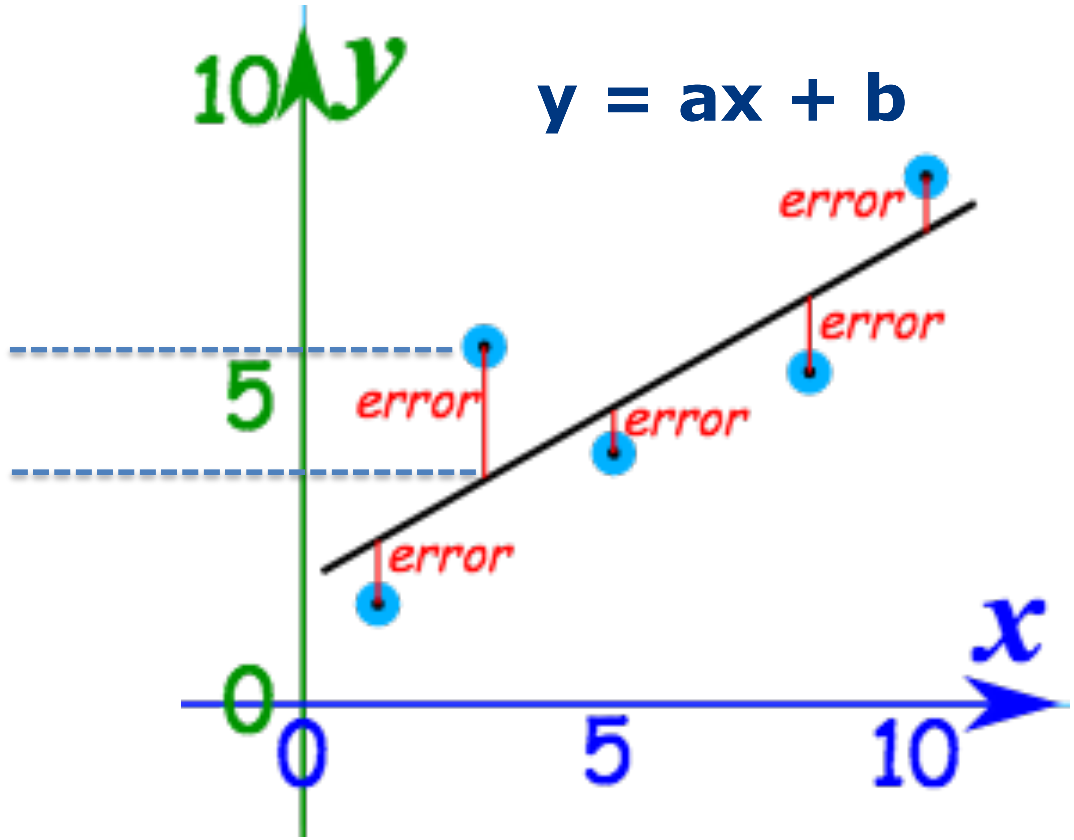
Linear least squares

- Goal: minimizing the sum of the square of the errors

$$\min \sum_{i=0}^{i=N} (y_i - y_i')^2$$

y_i: observed value

y_i': predicted value



Linear least squares

- Goal: minimizing the sum of the square of the errors

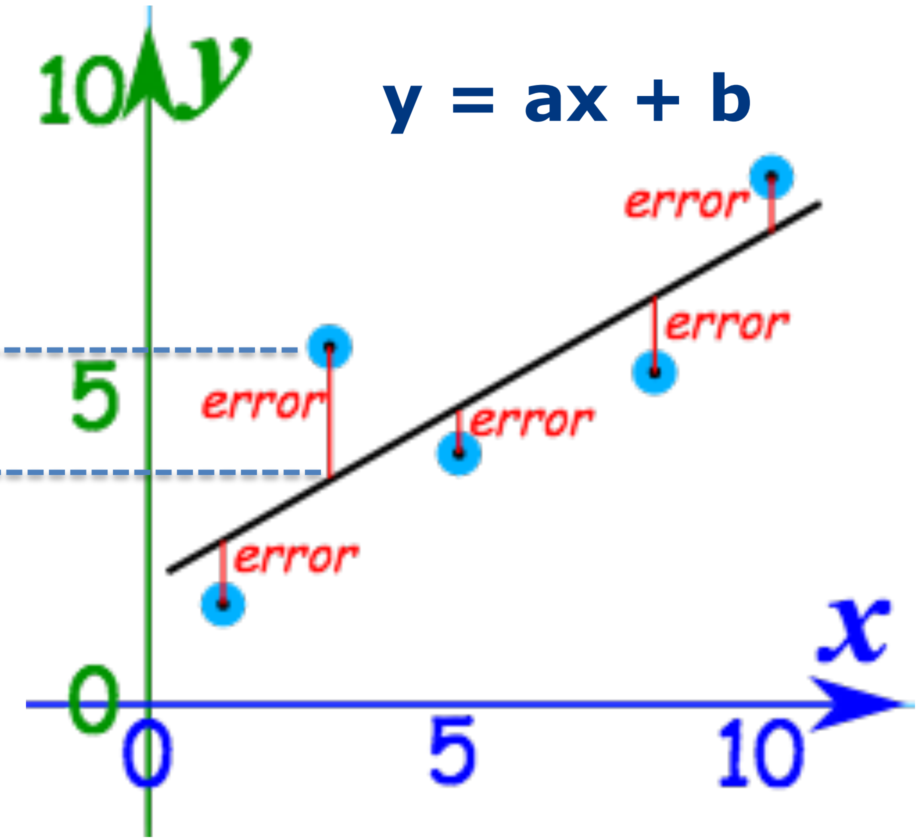
$$\min \sum_{i=0}^{i=N} (y_i - y_i')^2$$

y_i: observed value

y_i': predicted value

$$y_i' = ax_i + b$$

$$\sum_{i=0}^{i=N} [y_i - (ax_i + b)]^2$$



Linear least squares

- Goal: minimizing the sum of the square of the errors

$$\min \sum_{i=0}^{i=N} [y_i - (ax_i + b)]^2$$

Variables are *a* an *b*

Linear least squares

- Goal: minimizing the sum of the square of the errors

$$\min \sum_{i=0}^{i=N} [y_i - (ax_i + b)]^2$$ Variables are *a* an *b*

- Equivalent to solving the overdetermined linear system

$$y_1 - (ax_1 + b) = 0$$
$$y_2 - (ax_2 + b) = 0$$
$$y_3 - (ax_3 + b) = 0$$
$$\ldots$$

$$N \geq 2$$

Linear least squares

- Goal: minimizing the sum of the square of the errors

$$\mathbf{min} \sum_{i=0}^{i=N} [y_i - (ax_i + b)]^2$$ Variables are *a* an *b*

- Equivalent to solving the overdetermined linear system

$$y_1 - (ax_1 + b) = 0$$
$$y_2 - (ax_2 + b) = 0 \qquad N \geq 2$$
$$y_3 - (ax_3 + b) = 0$$
$$\ldots$$

- Solve it

$$A\mathbf{x} = \mathbf{b} \Rightarrow A^T A \mathbf{x} = A^T \mathbf{b} \Rightarrow \mathbf{x} = (A^T A)^{-1} A^T b$$

Other methods

- Row transformation w/o matrix notation
- Gaussian elimination
- Direct method $\mathbf{x} = A^{-1} b$
- Many other methods
  - LU decomposition

$$\begin{bmatrix} A00 & A01 & A02 \\ A10 & A11 & A12 \\ A20 & A21 & A22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L10 & 1 & 0 \\ L20 & L21 & 1 \end{bmatrix} \begin{bmatrix} U00 & U01 & U02 \\ 0 & U11 & U12 \\ 0 & 0 & U22 \end{bmatrix}$$

  - Cholesky decomposition

$$\mathbf{A} = \mathbf{L}\mathbf{L}^T = \begin{pmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \begin{pmatrix} L_{11} & L_{21} & L_{31} \\ 0 & L_{22} & L_{32} \\ 0 & 0 & L_{33} \end{pmatrix}$$

Assignment (part 2)

- System of linear equations
- Linear least squares
  - Use any programming language/tools
  - Use any third party libraries/software

Next Lecture

Part 3: Eigen values/vectors, singular value decomposition