# Lecture
# **Epipolar Geometry**

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# Today's Agenda

- Review Camera calibration
- Epipolar geometry

# Review of Camera Calibration

- Camera calibration
  - Recovering $K$
  - Recovering $R$ and $\mathbf{t}$

$$\mathbf{p} = M\mathbf{P}$$
$$= K \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \mathbf{P}$$

Internal (intrinsic) parameters

External (extrinsic) parameters

# Review of Camera Calibration

- How many parameters to recover?
  - 5 intrinsic parameters
    - 2 for focal length
    - 2 for offset
    - 1 for skewness
  - 6 extrinsic parameters
    - 3 for rotation
    - 3 for translation

$$\mathbf{p} = M\mathbf{P}$$
$$= \boxed{K} \boxed{\begin{bmatrix} R & \mathbf{t} \end{bmatrix}} \mathbf{P}$$

$$K = \begin{bmatrix} f_x & s & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} \mathbf{r}_1^T \\ \mathbf{r}_2^T \\ \mathbf{r}_3^T \end{bmatrix}, \quad \mathbf{t} = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

# Review of Camera Calibration

- Parameters to recover: 11
- Corresponding 3D-2D point pairs
  - Each 3D-2D point pair -> 2 constraints
  - 11 unknown -> 6 point correspondence
  - Use more to handle noisy data

$$\mathbf{p}_i = \begin{bmatrix} u_i \\ v_i \end{bmatrix} = M\mathbf{P}_i = \begin{bmatrix} \frac{\mathbf{P}_i^T \mathbf{m}_1}{\mathbf{P}_i^T \mathbf{m}_3} \\ \\ \frac{\mathbf{P}_i^T \mathbf{m}_2}{\mathbf{P}_i^T \mathbf{m}_3} \end{bmatrix} \Rightarrow \begin{matrix} \mathbf{P}_i^T \mathbf{m}_1 - u_i(\mathbf{P}_i^T \mathbf{m}_3) = 0 \\ \mathbf{P}_i^T \mathbf{m}_2 - v_i(\mathbf{P}_i^T \mathbf{m}_3) = 0 \end{matrix}$$

# Review of Camera Calibration

- Parameters to recover: 11
- Corresponding 3D-2D point pairs: >= 6
- How to solve it?
  - $\mathbf{m} = 0$ is always a trivial solution
  - If $\mathbf{m} \neq 0$ is a solution, then any $k * \mathbf{m}$ is also a solution

$$\begin{bmatrix} \mathbf{P}_1^T & 0^T & -u_1\mathbf{P}_1^T \\ 0^T & \mathbf{P}_1^T & -v_1\mathbf{P}_1^T \\ & \vdots & \\ \mathbf{P}_n^T & 0^T & -u_n\mathbf{P}_n^T \\ 0^T & \mathbf{P}_n^T & -v_n\mathbf{P}_n^T \end{bmatrix} \begin{bmatrix} \mathbf{m}_1^T \\ \mathbf{m}_2^T \\ \mathbf{m}_3^T \end{bmatrix} = P\mathbf{m} = 0$$

# Review of Camera Calibration

- Parameters to recover: 11
- Corresponding 3D-2D point pairs: >= 6
- How to solve it?
  - $\mathbf{m} = 0$ is always a trivial solution
  - If $\mathbf{m} \neq 0$ is a solution, then any $k * \mathbf{m}$ is also a solution
  - Constrained optimization

$$\begin{bmatrix} \mathbf{P}_1^T & 0^T & -u_1\mathbf{P}_1^T \\ 0^T & \mathbf{P}_1^T & -v_1\mathbf{P}_1^T \\ & \vdots & \\ \mathbf{P}_n^T & 0^T & -u_n\mathbf{P}_n^T \\ 0^T & \mathbf{P}_n^T & -v_n\mathbf{P}_n^T \end{bmatrix} \begin{bmatrix} \mathbf{m}_1^T \\ \mathbf{m}_2^T \\ \mathbf{m}_3^T \end{bmatrix} = P\mathbf{m} = 0 \quad \Rightarrow \quad \begin{aligned} &\underset{\mathbf{m}}{\text{minimize}} && \|P\mathbf{m}\|^2 \\ &\text{subject to} && \|\mathbf{m}\|^2 = 1 \end{aligned}$$

# Review of Camera Calibration

- Solved using SVD

$$P\mathbf{m} = 0$$

SVD decomposition of P

$$U_{2n\times 12}\, D_{12\times 12}\, V^{T}_{12\times 12}$$

Last column of V gives $\mathbf{m}$

(Why? See page 593 of Hartley & Zisserman. Multiple view geometry in computer vision)

# Review of Camera Calibration

Intrinsic parameters:

$$\rho = \pm \frac{1}{\|\mathbf{a}_3\|}$$

$$c_x = \rho^2 (\mathbf{a}_1 \cdot \mathbf{a}_3)$$

$$c_y = \rho^2 (\mathbf{a}_2 \cdot \mathbf{a}_3)$$

$$\cos\theta = -\frac{(\mathbf{a}_1 \times \mathbf{a}_3) \cdot (\mathbf{a}_2 \times \mathbf{a}_3)}{\|\mathbf{a}_1 \times \mathbf{a}_3\| \cdot \|\mathbf{a}_2 \times \mathbf{a}_3\|}$$

$$\alpha = \rho^2 \|\mathbf{a}_1 \times \mathbf{a}_3\| \sin\theta$$

$$\beta = \rho^2 \|\mathbf{a}_2 \times \mathbf{a}_3\| \sin\theta$$

Extrinsic parameters:

$$\mathbf{r}_1 = \frac{\mathbf{a}_2 \times \mathbf{a}_3}{\|\mathbf{a}_2 \times \mathbf{a}_3\|}$$

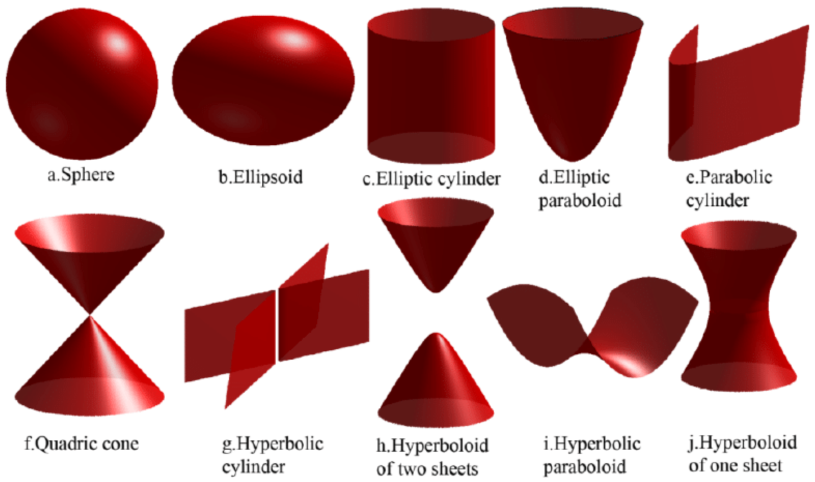
$$\mathbf{r}_3 = \rho \mathbf{a}_3$$

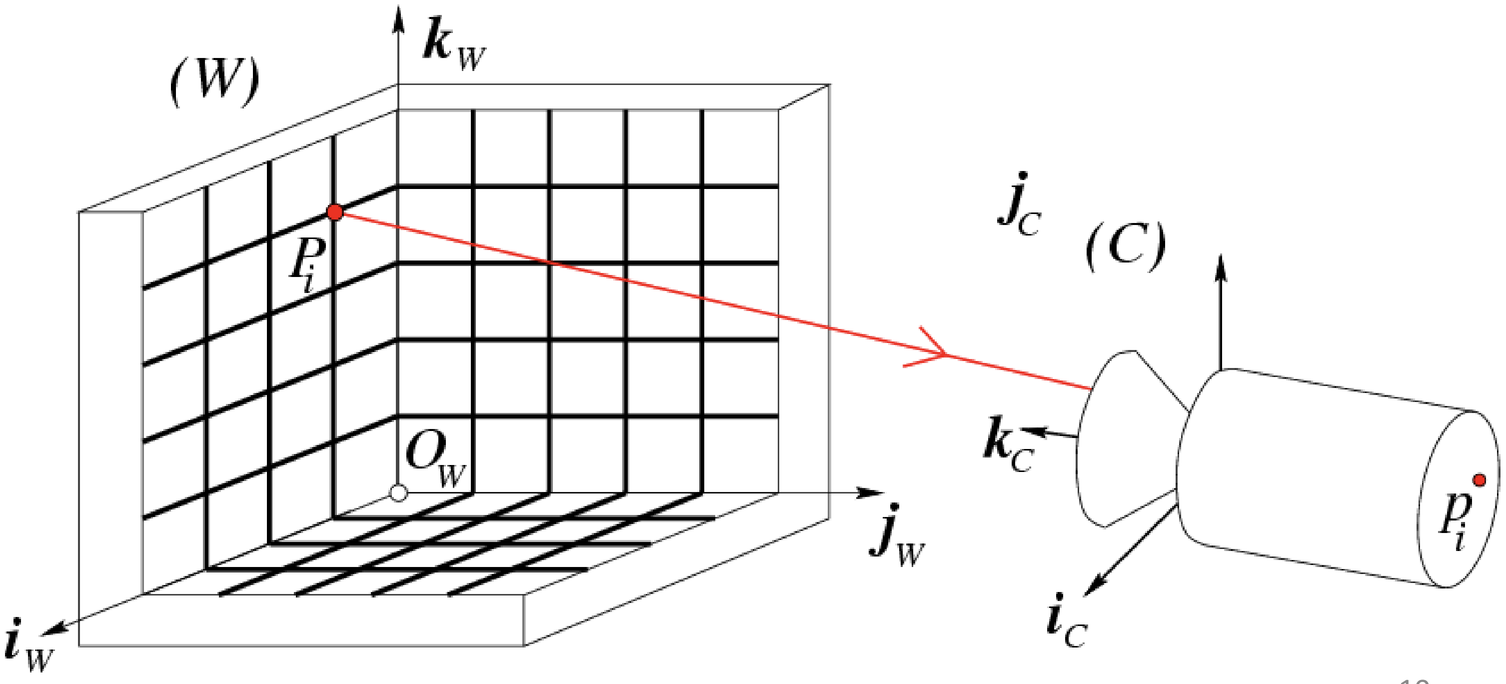
$$\mathbf{r}_2 = \mathbf{r}_3 \times \mathbf{r}_1$$

$$\mathbf{t} = \rho K^{-1} \mathbf{b}$$

# Review of Camera Calibration

- Not always solvable
  - $\{P_i\}$ cannot lie on the same plane
  - $\{P_i\}$ cannot lie on the intersection curve of two quadric surfaces



# Quiz

Which of the following will change the camera intrinsic matrix?

(a) When zooming in.

(b) When rotating the camera around its local origin.

(c) When changing the resolution of the image.

(d) When the camera is moved.

# Quiz

Which of the following will change the camera intrinsic matrix?

(a) When zooming in. $[f_x, f_y]$

(b) When rotating the camera around its local origin. $R$

(c) When changing the resolution of the image. $[c_x, c_y]$

(d) When the camera is moved. $\mathbf{t}$

$$K = \begin{bmatrix} f_x & s & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix}$$

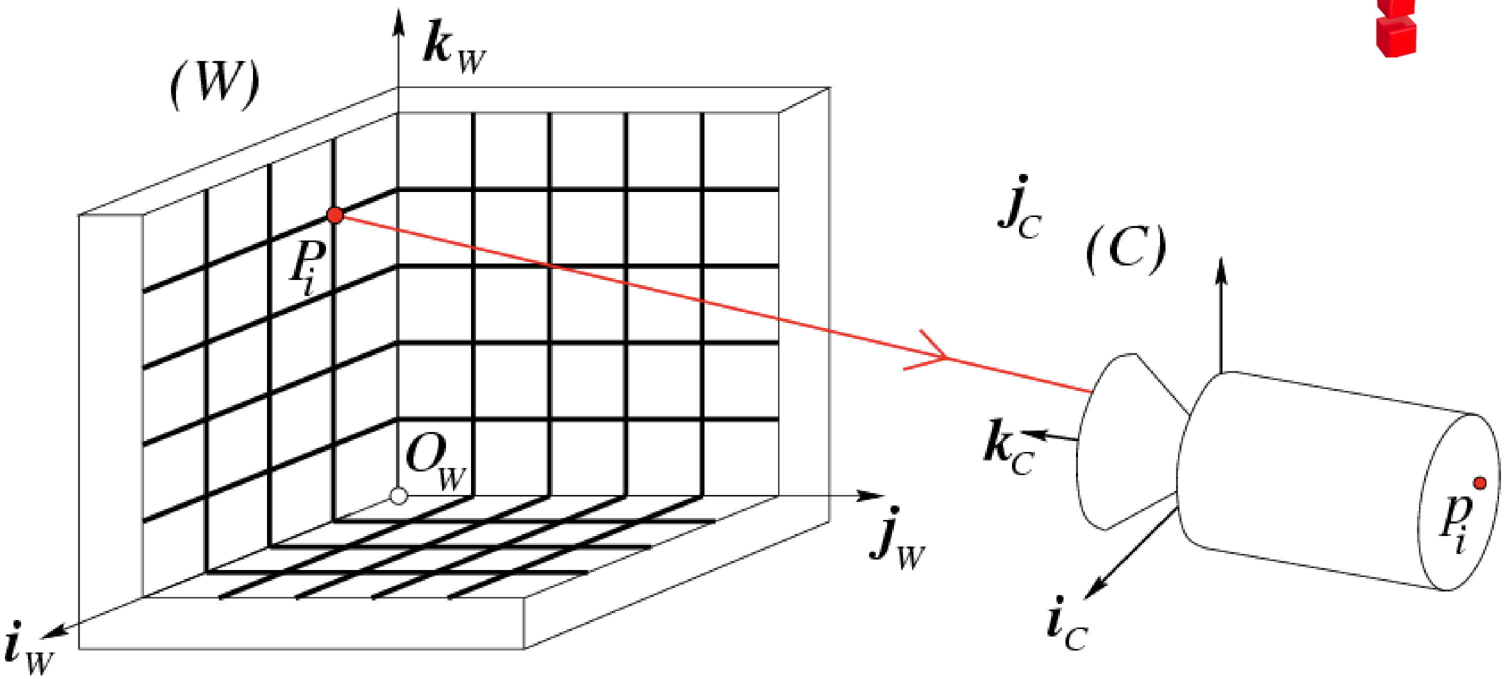
# Today’s Agenda

- Review Camera calibration
- Epipolar geometry

# Recovering 3D Geometry

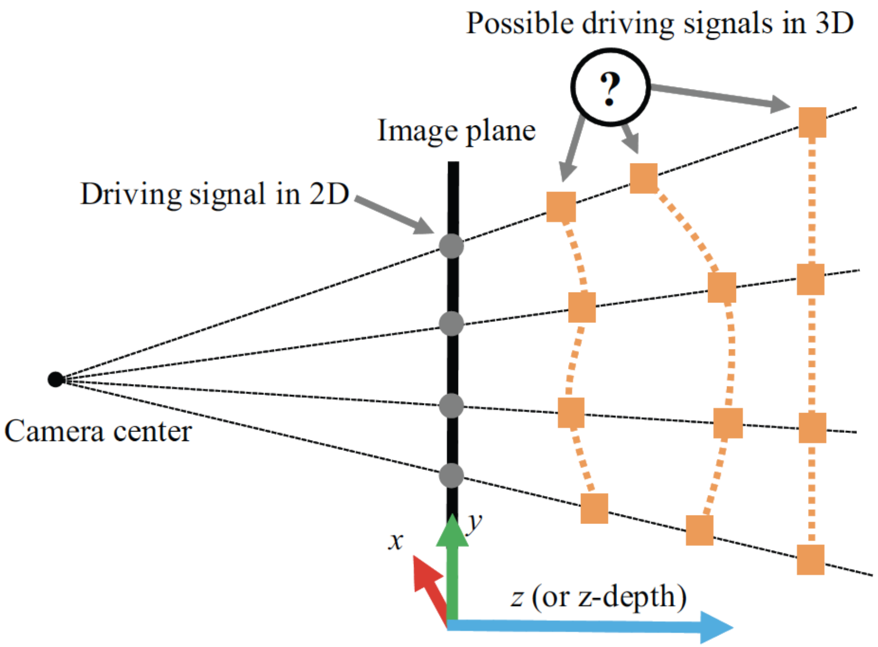
- Camera calibration from a single view
  - Camera intrinsic parameters
  - Camera orientation
  - Camera translation

Sufficient to recover some 3D geometry from a single image?



# Recovering 3D Geometry

- Camera calibration from a single view
- Recover 3D geometry from a single view?
  – No: due to ambiguity of 3D -> 2D mapping

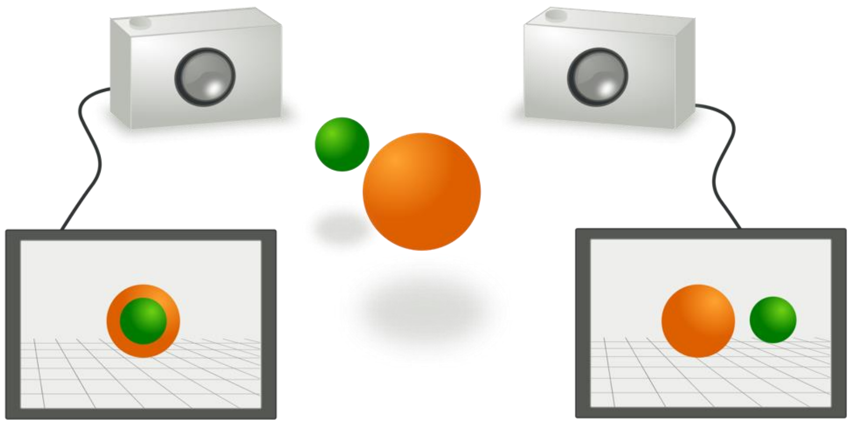



# Core Problems in Recovering 3D Geometry

- **Image correspondences**: find the corresponding points in two or more images – **lecture, code, lab exercise**
- **Calibration**: given corresponding points in images, recover the relation of the cameras. **Epipolar geometry**
- **Recover scene geometry**: reconstruct coordinates of 3D points from corresponding image pixels – **next lecture**

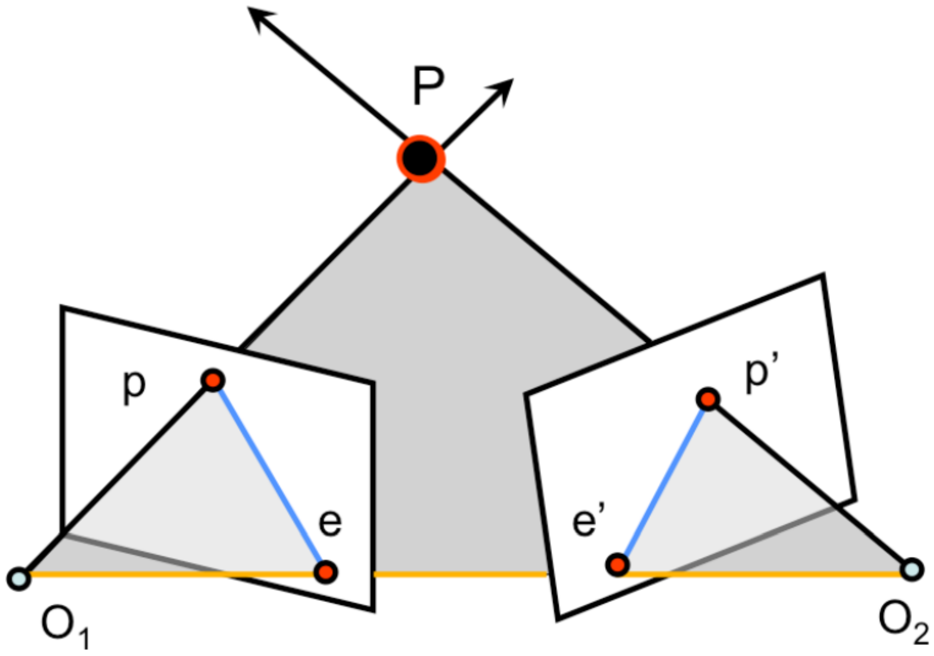
# Epipolar Geometry

- Camera model
  - Relate 3D points and corresponding images points
- Epipolar geometry
  - Geometric relations between the corresponding image points
  - Used to recover the relation of the cameras



# Epipolar Geometry

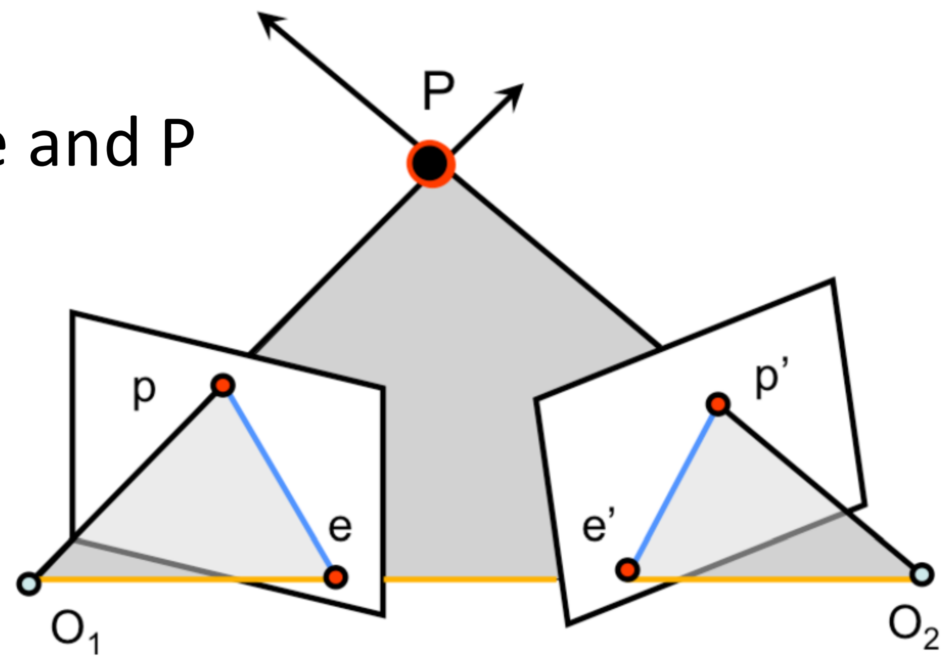
- Baseline
  - The line between the two camera centers $O_1$ and $O_2$




The general setup of epipolar geometry

# Epipolar Geometry

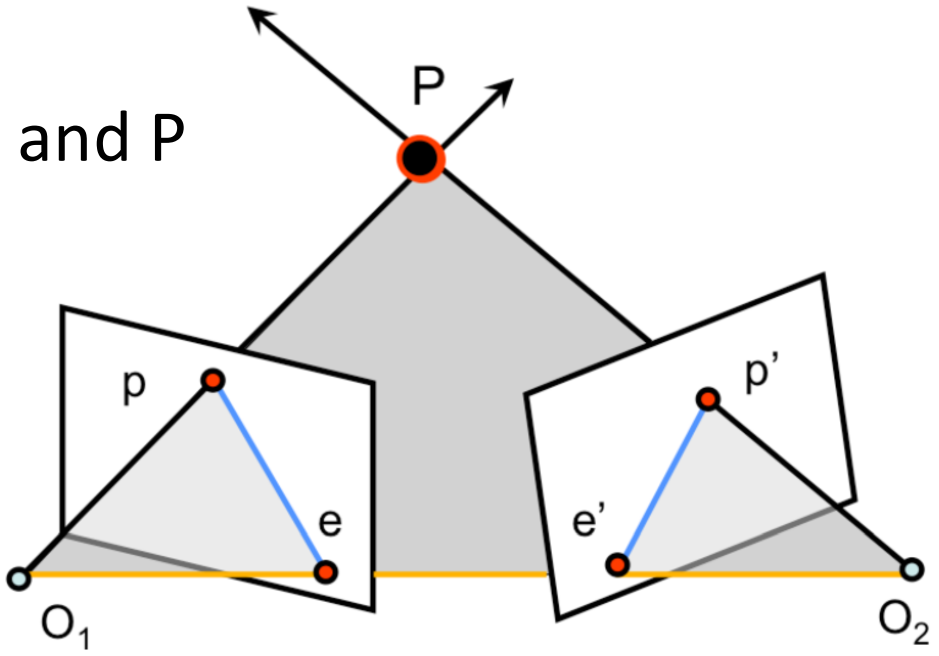
- Baseline
  - The line between the two camera centers $O_1$ and $O_2$
- Epipolar plane
  - Defined by P, $O_1$, and $O_2$; contains baseline and P




The general setup of epipolar geometry

# Epipolar Geometry

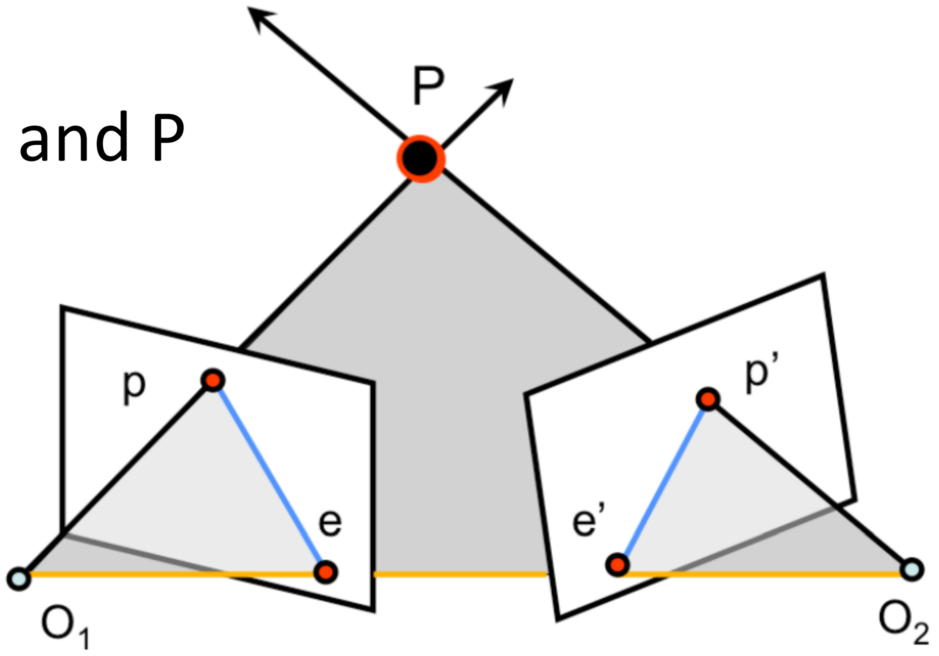
- Baseline
  - The line between the two camera centers $O_1$ and $O_2$
- Epipolar plane
  - Defined by P, $O_1$, and $O_2$; contains baseline and P
- Epipoles
  - ∩ of baseline and image plane: e and e’
  - Projection of the other camera center




The general setup of epipolar geometry

# Epipolar Geometry

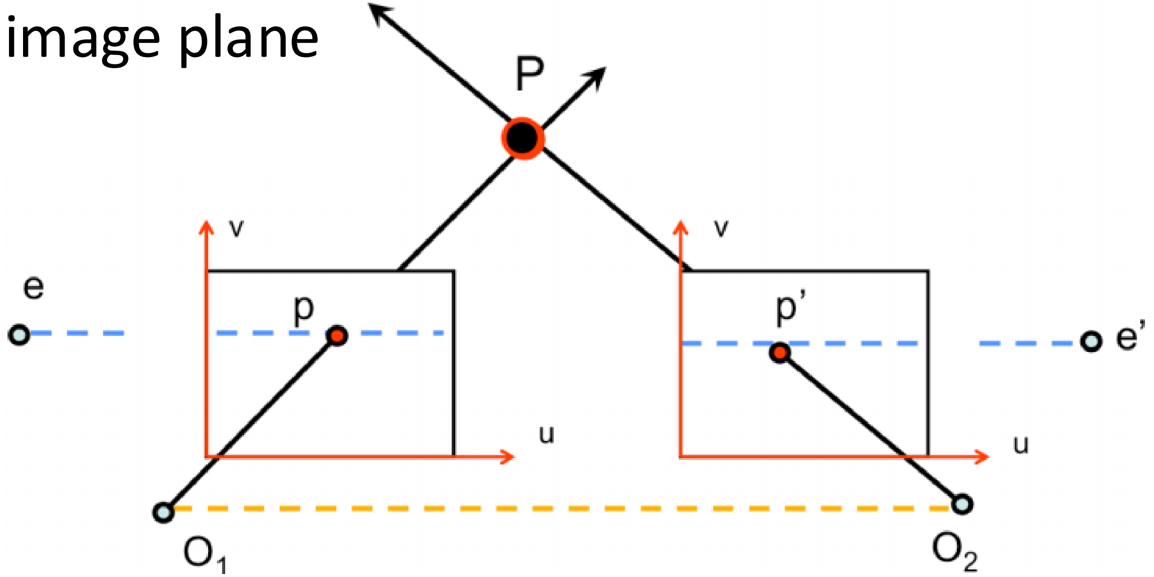
- Baseline
  - The line between the two camera centers $O_1$ and $O_2$
- Epipolar plane
  - Defined by P, $O_1$, and $O_2$; contains baseline and P
- Epipoles
  - ∩ of baseline and image plane
  - Projection of the other camera center
- Epipolar lines
  - ∩ of epipolar plane with the image plane




The general setup of epipolar geometry

# Epipolar Geometry

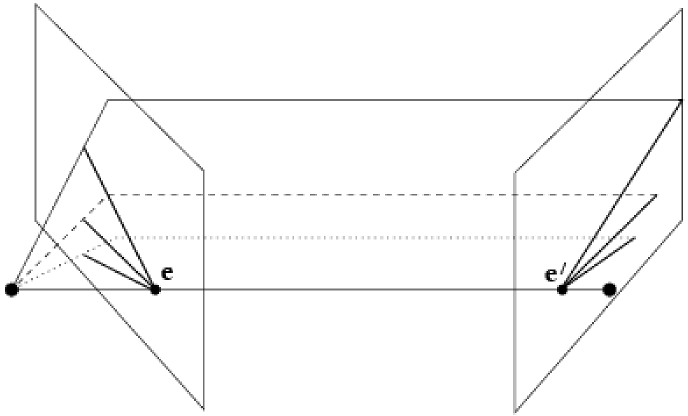
- Examples
  - Parallel image planes (a special case)
    - Baseline is parallel to the image plane
    - Baseline intersects the image plane at infinity → epipoles are at infinity
    - Epipolar lines are parallel to U-axis of image plane

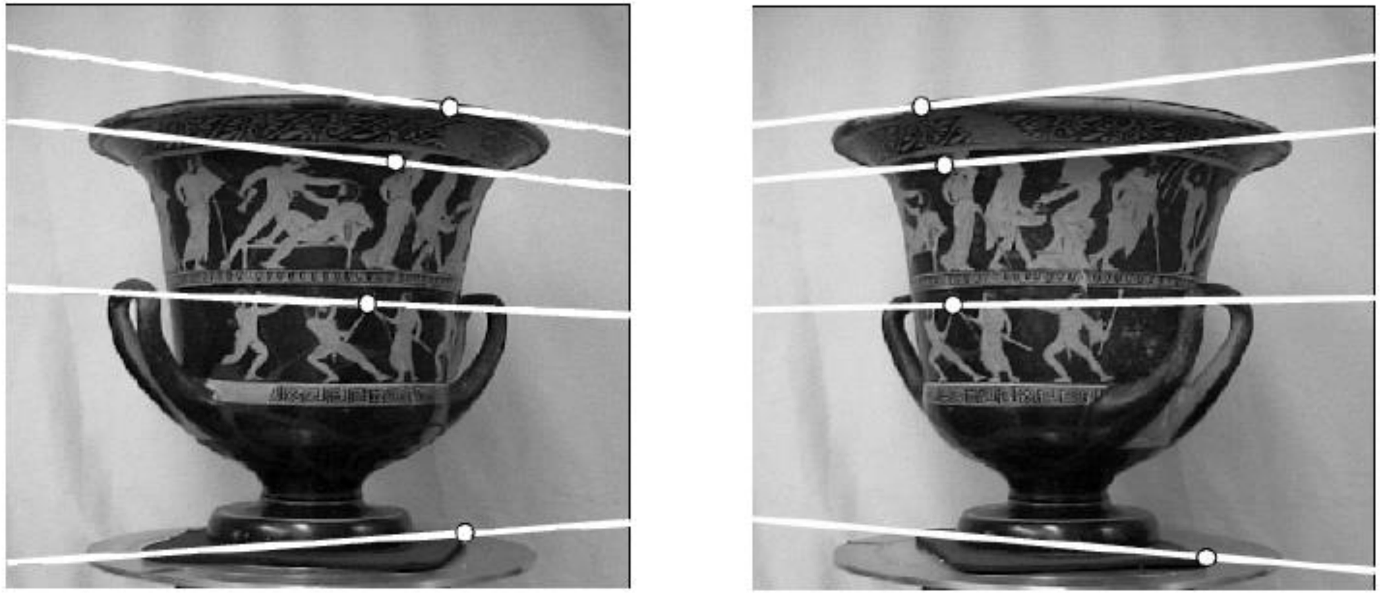


# Epipolar Geometry

- Examples
  - Converging image planes (most common case)
    - All epipolar lines intersect at the epipole

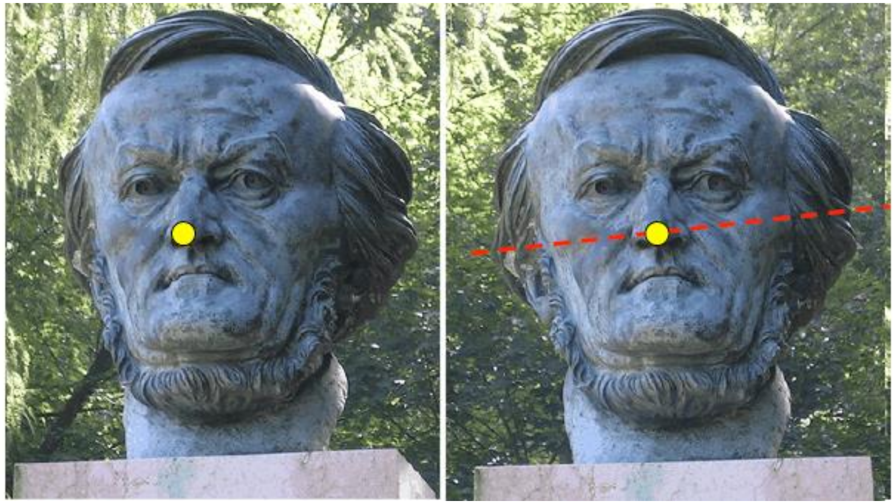
- all epipolar lines lie on epipolar planes
- all epipolar planes intersect at baseline
- base line intersect the image plane at epipole

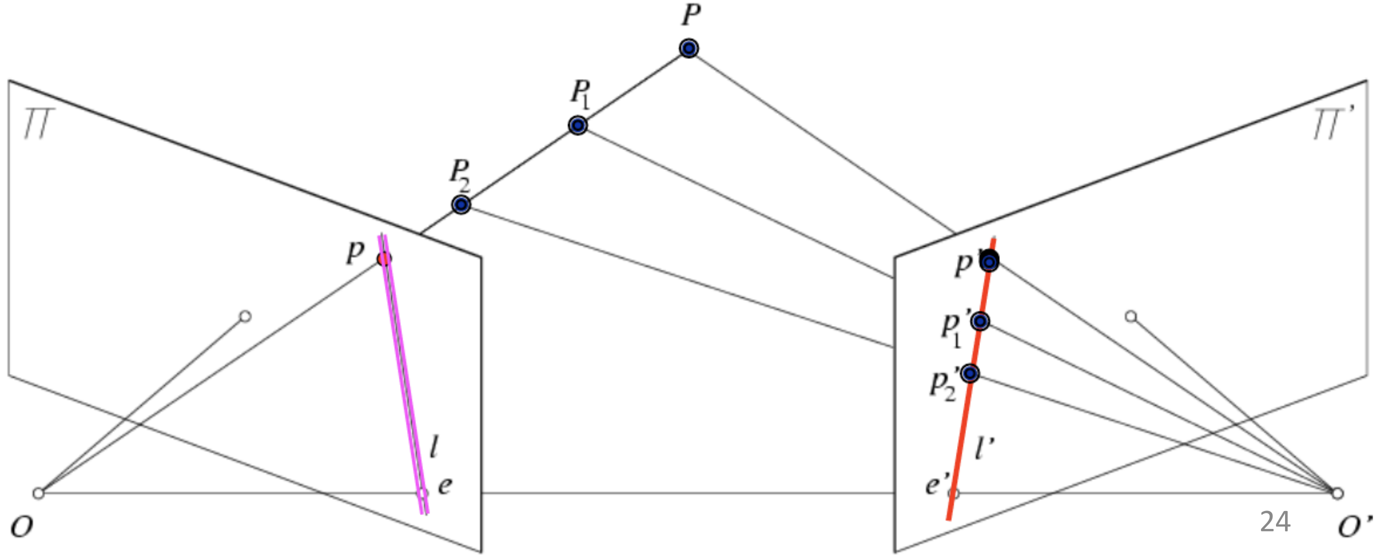


# Epipolar Constraint

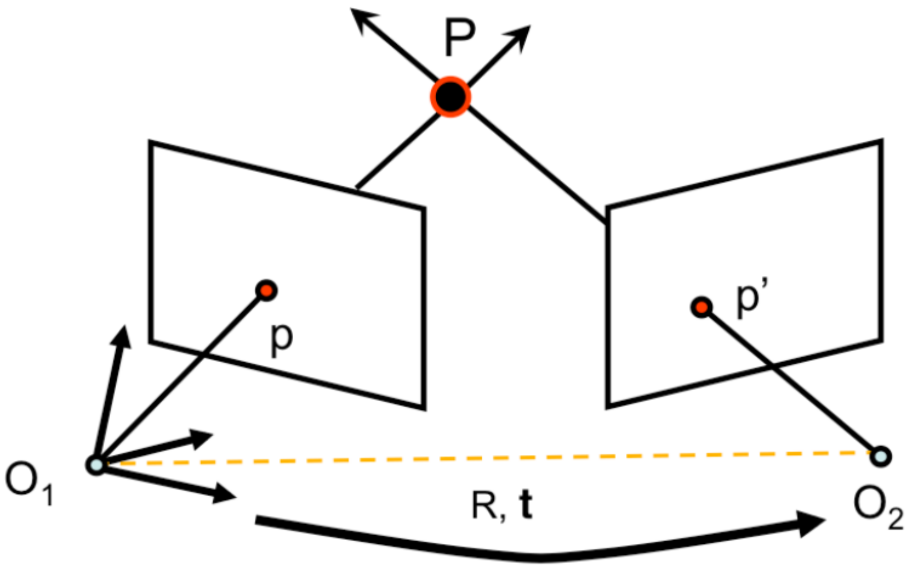
- Given a point on left image, what are the potential locations of the corresponding point on right image?
  - – have to lie on the corresponding epipolar line of the other image
- Model of the relation between corresponding image points





# Epipolar Constraint

- The relationship between corresponding image points
  - The world reference system aligned with the left camera
  - The right camera has orientation $R$ and offset $\mathbf{t}$




Camera projection matrices

Left camera

$$M = K[I \;\; 0]$$

$$\mathbf{p} = M\mathbf{P} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

Right camera

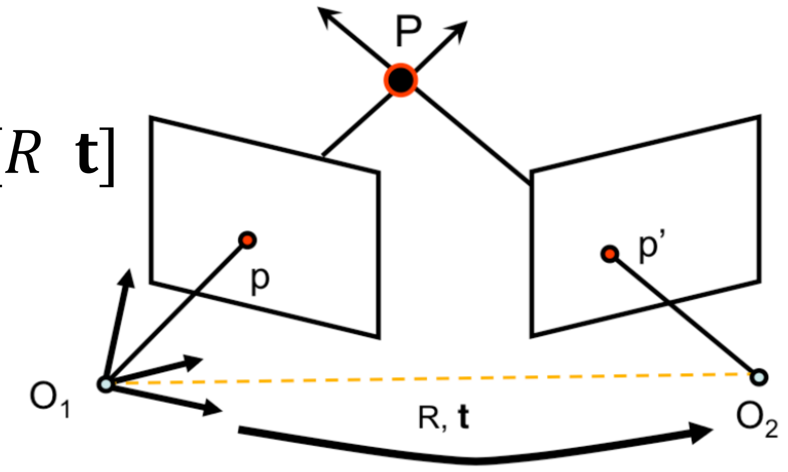
$$M' = K'[R \;\; \mathbf{t}]$$

$$\mathbf{p}' = M'\mathbf{P} = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix}$$

# Epipolar Constraint

- The relationship between corresponding image points
  - Canonical cameras ($K = K' = I$)

$$M = K[I \;\; 0] \rightarrow M = [I \;\; 0] \qquad M' = K'[R \;\; \mathbf{t}] \rightarrow M' = [R \;\; \mathbf{t}]$$

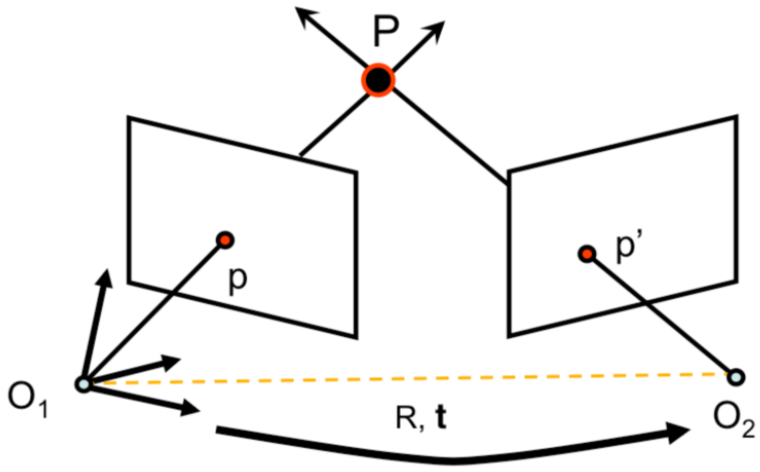



**p'** in world coordinate system

# Epipolar Constraint

- The relationship between corresponding image points
  - Canonical cameras ($K = K' = I$)

$$M = [I \;\; 0] \qquad M' = [R \;\; \mathbf{t}]$$




**p'** in world coordinate system $\qquad R^T(\mathbf{p}' - \mathbf{t})$

**$O_2$** in world coordinate system

# Epipolar Constraint

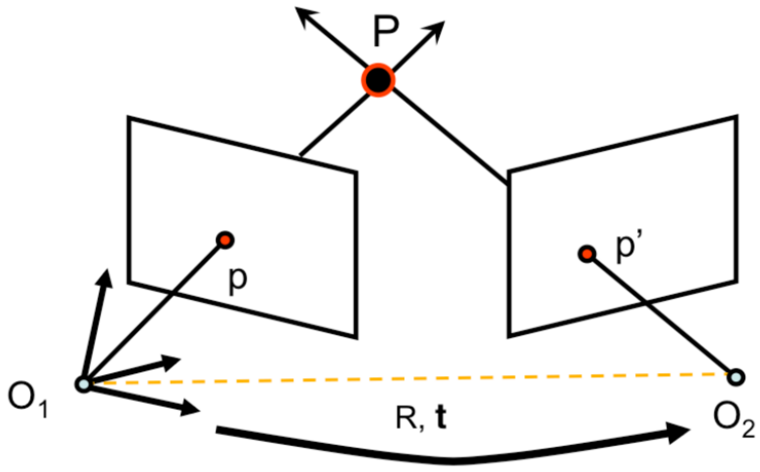
- The relationship between corresponding image points
  - Canonical cameras ($K = K' = I$)

    $M = [I \;\; 0] \qquad M' = [R \;\; \mathbf{t}]$

**p'** in world coordinate system $\quad R^T(\mathbf{p}' - \mathbf{t})$

**$\mathbf{O_2}$** in world coordinate system $\quad R^T(\mathbf{O_2} - \mathbf{t}) = -R^T\mathbf{t}$

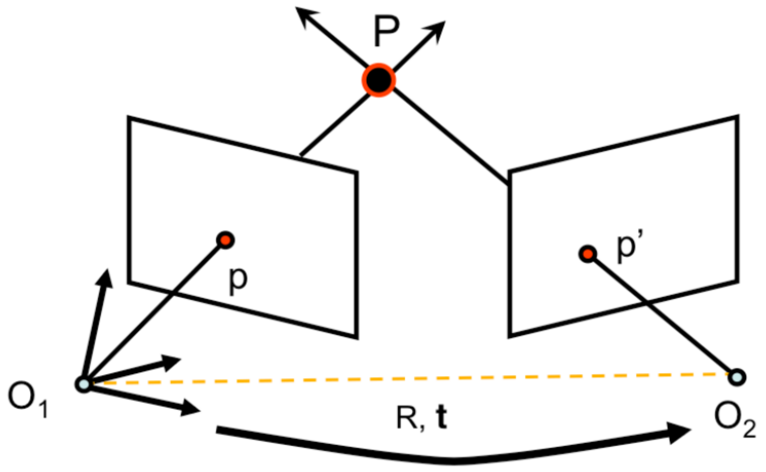
Normal of the epipolar plane



# Epipolar Constraint

- The relationship between corresponding image points
  - Canonical cameras ($K = K' = I$)

$$M = [I \;\; 0] \qquad M' = [R \;\; \mathbf{t}]$$




**p'** in world coordinate system $R^T(\mathbf{p}' - \mathbf{t})$

**O₂** in world coordinate system $R^T(\mathbf{O_2} - \mathbf{t}) = -R^T\mathbf{t}$

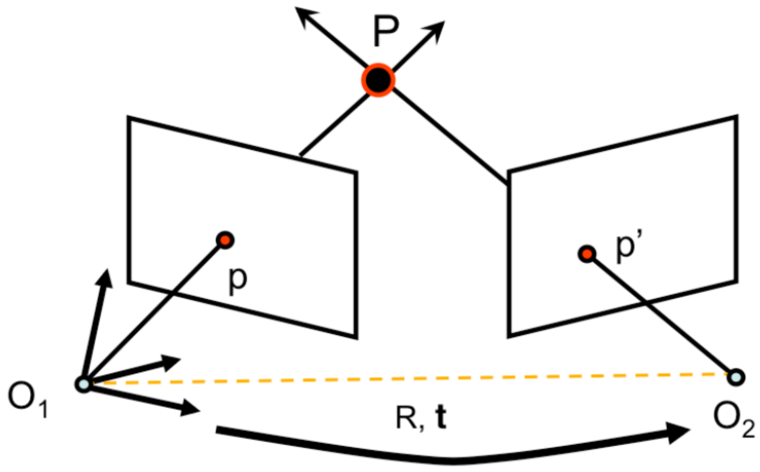
Normal of the epipolar plane $R^T\mathbf{t} \times [R^T(\mathbf{p}' - \mathbf{t})] = R^T(\mathbf{t} \times \mathbf{p}')$

$\mathbf{O_1p}$ lies in the epipolar plane: so the dot product of the normal of the Epipolar plane and $\mathbf{O_1p}$ is 0

# Epipolar Constraint

- The relationship between corresponding image points
  - Canonical cameras ($K = K' = I$)

$$M = [I \;\; 0] \qquad M' = [R \;\; \mathbf{t}]$$




**p'** in world coordinate system $R^T(\mathbf{p}' - \mathbf{t})$

**O₂** in world coordinate system $R^T(\mathbf{O_2} - \mathbf{t}) = -R^T\mathbf{t}$

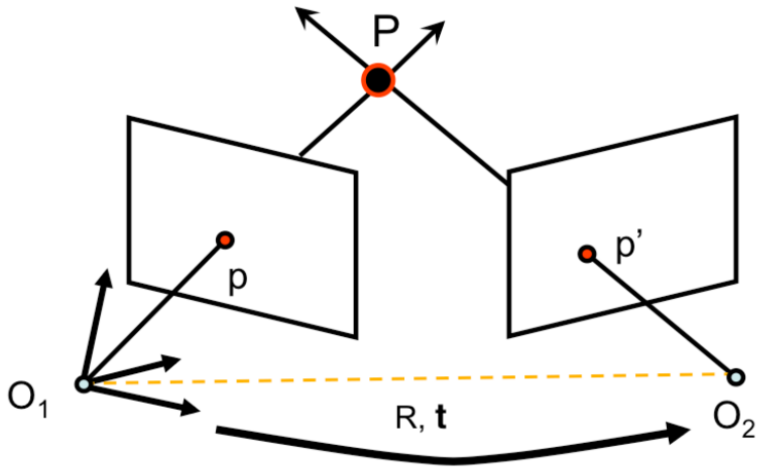
Normal of the epipolar plane $R^T\mathbf{t} \times [R^T(\mathbf{p}' - \mathbf{t})] = R^T(\mathbf{t} \times \mathbf{p}')$

**$O_1$p** lies in the epipolar plane $[R^T(\mathbf{t} \times \mathbf{p}')]^T\mathbf{p} = 0$

# Epipolar Constraint

- The relationship between corresponding image points
  - Canonical cameras ($K = K' = I$)

$$M = [I \;\; 0] \qquad M' = [R \;\; \mathbf{t}]$$




**p'** in world coordinate system $\quad R^T(\mathbf{p}' - \mathbf{t})$

**$O_2$** in world coordinate system $\quad R^T(\mathbf{O}_2 - \mathbf{t}) = -R^T\mathbf{t}$

Normal of the epipolar plane $\quad R^T\mathbf{t} \times [R^T(\mathbf{p}' - \mathbf{t})] = R^T(\mathbf{t} \times \mathbf{p}')$

**$O_1$p** lies in the epipolar plane $\quad [R^T(\mathbf{t} \times \mathbf{p}')]^T\mathbf{p} = 0 \Rightarrow (\mathbf{t} \times \mathbf{p}')^T R\mathbf{p} = 0$

# Epipolar Constraint

- The relationship between corresponding image points
  - Canonical cameras ($K = K' = I$)

$$M = [I \;\; 0] \qquad M' = [R \;\; \mathbf{t}]$$

Cross product as matrix-vector multiplication

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -\mathbf{a}_z & \mathbf{a}_y \\ \mathbf{a}_z & 0 & -\mathbf{a}_x \\ -\mathbf{a}_y & \mathbf{a}_x & 0 \end{bmatrix} \begin{bmatrix} \mathbf{b}_x \\ \mathbf{b}_y \\ \mathbf{b}_z \end{bmatrix} = [\mathbf{a}_\times]\mathbf{b}$$

$[\mathbf{a}_\times]^T = -[\mathbf{a}_\times]$

**p'** in world coordinate system $\quad R^T(\mathbf{p}' - \mathbf{t})$

**$\mathbf{O_2}$** in world coordinate system $\quad R^T(\mathbf{O_2} - \mathbf{t}) = -R^T\mathbf{t}$

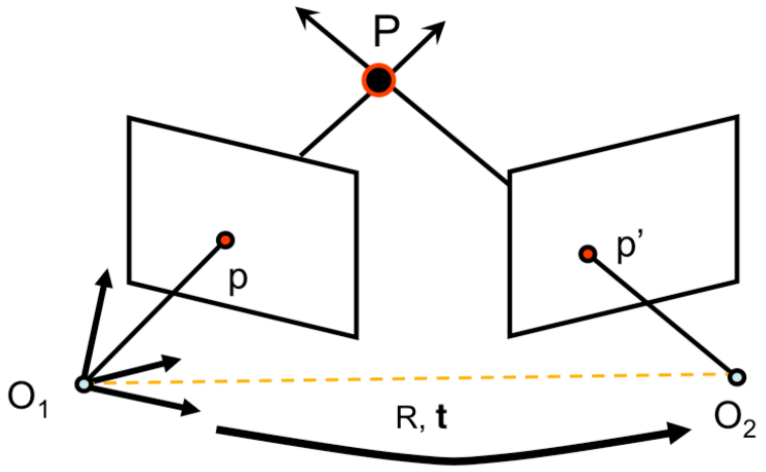
Normal of the epipolar plane $\quad R^T\mathbf{t} \times [R^T(\mathbf{p}' - \mathbf{t})] = R^T(\mathbf{t} \times \mathbf{p}')$

**$\mathbf{O_1p}$** lies in the epipolar plane $\quad [R^T(\mathbf{t} \times \mathbf{p}')]^T\mathbf{p} = 0 \Rightarrow (\mathbf{t} \times \mathbf{p}')^T R\mathbf{p} = 0 \Rightarrow ([\mathbf{t}_\times]\,\mathbf{p}')^T R\mathbf{p} = 0$

# Epipolar Constraint

- The relationship between corresponding image points
  - Canonical cameras ($K = K' = I$)

$$M = [I \;\; 0] \qquad M' = [R \;\; \mathbf{t}]$$




**p'** in world coordinate system $\quad R^T(\mathbf{p}' - \mathbf{t})$

**$\mathbf{O_2}$** in world coordinate system $\quad R^T(\mathbf{O_2} - \mathbf{t}) = -R^T\mathbf{t}$

Normal of the epipolar plane $\quad R^T\mathbf{t} \times [R^T(\mathbf{p}' - \mathbf{t})] = R^T(\mathbf{t} \times \mathbf{p}')$

**$\mathbf{O_1 p}$** lies in the epipolar plane $\quad [R^T(\mathbf{t} \times \mathbf{p}')]^T\mathbf{p} = 0 \Rightarrow (\mathbf{t} \times \mathbf{p}')^T R\mathbf{p} = 0 \Rightarrow ([\mathbf{t}_\times]\,\mathbf{p}')^T R\mathbf{p} = 0$

$$\Rightarrow \mathbf{p}'^T[\mathbf{t}_\times]R\mathbf{p} = 0$$

# Epipolar Constraint

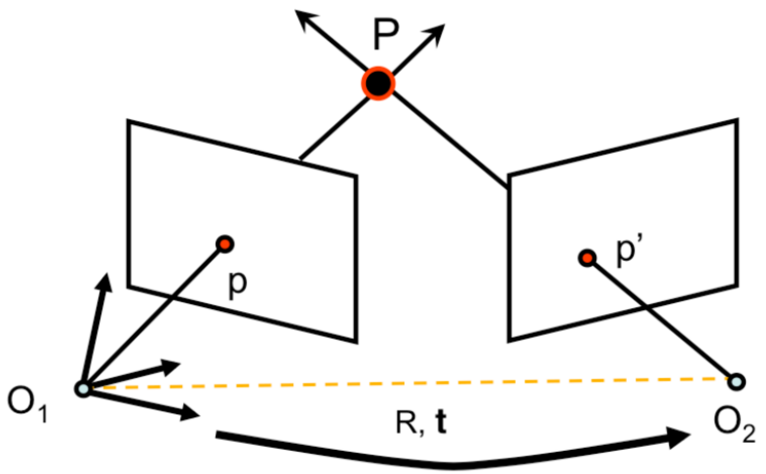
- Essential matrix
  - Establish constraints between matching image points
  - Determine relative position and orientation of two cameras
  - 5 degrees of freedom ($R$: 3, $\mathbf{t}$: 3, but scale is not known)

$$\mathbf{p}'^T [\mathbf{t}_\times] R \mathbf{p} = 0$$

$$E = [\mathbf{t}_\times] R$$

$$\mathbf{p}'^T E \mathbf{p} = 0$$

Essential matrix



# Epipolar Constraint

- How to generalize Essential matrix?
  - Canonical cameras → general cameras

$$M = K[I \;\; 0]$$
$$M' = K'[R \;\; \mathbf{t}]$$

$K = K' = I$ →

$$M = [I \;\; 0]$$
$$M' = [R \;\; \mathbf{t}]$$

→

$$\mathbf{p} = M\mathbf{P} = [I \;\; 0]\mathbf{P}$$
$$\mathbf{p}' = M'\mathbf{P} = [R \;\; \mathbf{t}]\mathbf{P}$$

→

$$\mathbf{p}'^{T}E\mathbf{p} = 0$$
$$E = [\mathbf{t}_{\times}]R$$

$K \neq I, K' \neq I$ →

# Epipolar Constraint

- How to generalize Essential matrix?
  - Canonical cameras → general cameras

Canonical cameras: the image points in homogeneous coordinates are actually the 3D point expressed in the camera coordinate system

$M = K[I \;\; 0]$

$M' = K'[R \;\; \mathbf{t}]$

$K = K' = I$

$M = [I \;\; 0]$

$M' = [R \;\; \mathbf{t}]$

$\mathbf{p} = M\mathbf{P} = [I \;\; 0]\mathbf{P}$

$\mathbf{p}' = M'\mathbf{P} = [R \;\; \mathbf{t}]\mathbf{P}$

$\mathbf{p}'^T E \mathbf{p} = 0$

$E = [\mathbf{t}_\times]R$

$K \neq I, K' \neq I$

$\mathbf{p} = M\mathbf{P} = K[I \;\; 0]\mathbf{P}$

$\mathbf{p}' = M'\mathbf{P} = K'[R \;\; \mathbf{t}]\mathbf{P}$

$\mathbf{p} \rightarrow K^{-1}\mathbf{p}$

$\mathbf{p}' \rightarrow K'^{-1}\mathbf{p}'$

$\mathbf{p}'^T K'^{-T} E K^{-1} \mathbf{p} = 0$

$\mathbf{p}'^T F \mathbf{p} = 0$

$F = K'^{-T}[\mathbf{t}_\times]RK^{-1}$

# Epipolar Constraint

- Essential matrix vs. Fundamental matrix
  - Similarity
    - Both relate the matching image points
      - Encode epipolar geometry of two views & camera parameters
  - Differences
    - $E$ encodes only the camera extrinsic parameter
    - $F$ also encodes the intrinsic parameters

$$\mathbf{p}'^T E \mathbf{p} = 0$$

$$E = [\mathbf{t}_\times] R$$

Essential matrix

$$\mathbf{p}'^T F \mathbf{p} = 0$$

$$F = K'^{-T} [\mathbf{t}_\times] R K^{-1}$$

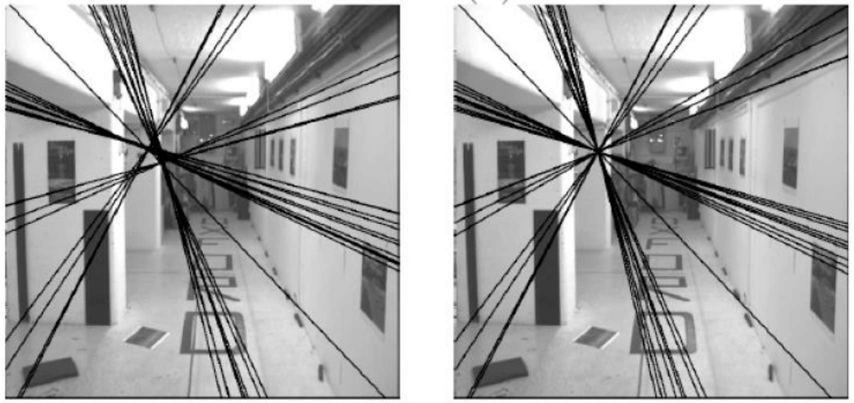
Fundamental matrix

# Epipolar Constraint

- Properties of the Fundamental matrix
  - 3 by 3
  - homogeneous (has scale ambiguity)
  - rank($F$) = 2
    - The potential matching point is located on a line
  - $F$ has 7 degrees of freedom

$$\mathbf{p}'^T F \mathbf{p} = 0 \qquad F = K'^{-T}[\mathbf{t}_{\times}]RK^{-1}$$

Fundamental matrix has rank 2 : $\det(\mathtt{F}) = 0$.



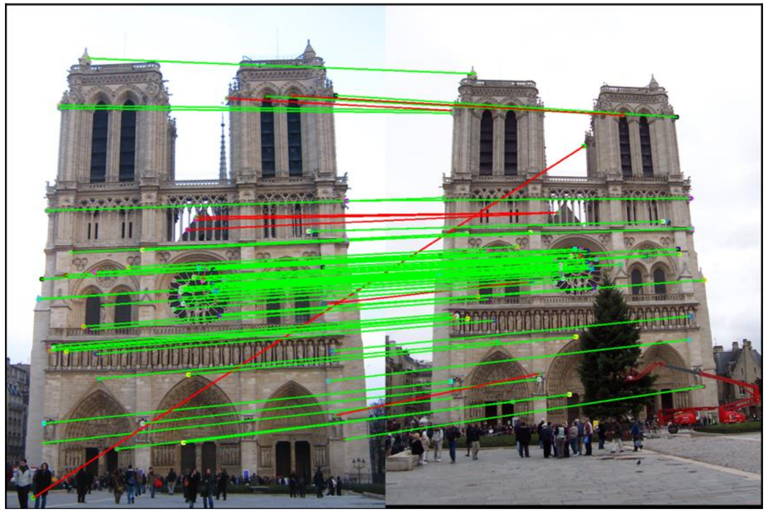
**Left :** Uncorrected F – epipolar lines are not coincident.

**Right :** Epipolar lines from corrected F.

# Epipolar Constraint

- Properties of the Fundamental matrix
- How is the fundamental matrix useful?
  - A 3D point's image in one image -> the epipolar line in the other image
    - Without knowing 3D location, camera intrinsic and extrinsic parameters
  - Powerful tool
    - Establishing reliable correspondences
    - Multi-view object/scene matching
    - Multi-view camera calibration

$$\mathbf{p}'^T F \mathbf{p} = 0 \qquad F = K'^{-T}[\mathbf{t}_{\times}]RK^{-1}$$

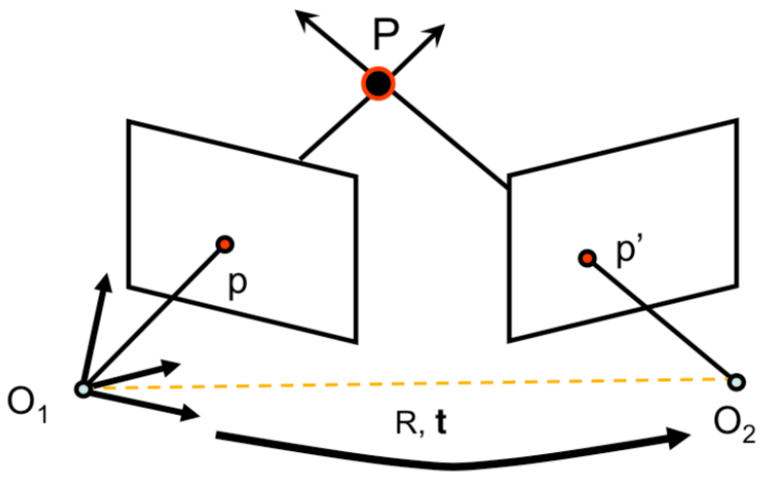


# Recovering Fundamental Matrix

- How to recover $F$?
  - From image correspondences

$\mathbf{p}'^T F \mathbf{p} = 0$

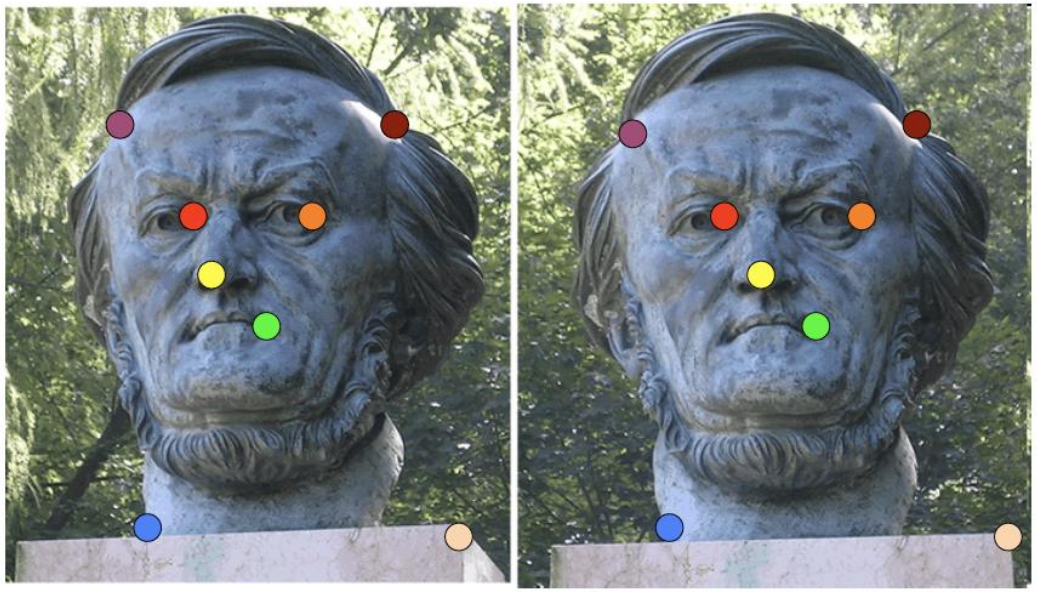
$F = K'^{-T}[\mathbf{t}_{\times}]RK^{-1}$



# Recovering Fundamental Matrix

- How to recover $F$?
  - From image correspondences
  - How many point pairs needed?

$$\mathbf{p}'^T F \mathbf{p} = 0 \qquad F = K'^{-T}[\mathbf{t}_{\times}]RK^{-1}$$



# Recovering Fundamental Matrix

- How to recover $F$?
  - From image correspondences
  - 8-point pairs required
    - Each point pair gives one equation
    - $F$ is known up to scale
      - The linear system is homogeneous

$$\begin{cases} \mathbf{p}_i = (u_i, v_i, 1) \\ \mathbf{p}_i' = (u_i', v_i', 1) \end{cases} \qquad \mathbf{p}'^T F \, \mathbf{p} = 0$$

$$\begin{bmatrix} u_i u_i' & v_i u_i' & u_i' & u_i v_i' & v_i v_i' & v_i' & u_i & v_i & 1 \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{bmatrix} = 0$$

# Recovering Fundamental Matrix

- How to recover $F$?
  - From image correspondences
  - 8-point pairs required
    - Each point pair gives one equation
    - $F$ is known up to scale
      - The linear system is homogeneous

$$\begin{cases} \mathbf{p}_i = (u_i, v_i, 1) \\ \mathbf{p}'_i = (u'_i, v'_i, 1) \end{cases} \qquad \mathbf{p}'^T F \, \mathbf{p} = 0$$

$$\begin{bmatrix} u_i u'_i & v_i u'_i & u'_i & u_i v'_i & v_i v'_i & v'_i & u_i & v_i & 1 \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{bmatrix} = 0$$

$F$ has 7 degrees of freedom

Are 7-point pairs sufficient?

# Recovering Fundamental Matrix

- 8-point algorithm

$$\begin{bmatrix} u_i u_i' & v_i u_i' & u_i' & u_i v_i' & v_i v_i' & v_i' & u_i & v_i & 1 \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{bmatrix} = 0$$

$$\begin{bmatrix} u_1 u_1' & v_1 u_1' & u_1' & u_1 v_1' & v_1 v_1' & v_1' & u_1 & v_1 & 1 \\ u_2 u_2' & v_2 u_2' & u_2' & u_2 v_2' & v_2 v_2' & v_2' & u_2 & v_2 & 1 \\ u_3 u_3' & v_3 u_3' & u_3' & u_3 v_3' & v_3 v_3' & v_3' & u_3 & v_3 & 1 \\ u_4 u_4' & v_4 u_4' & u_4' & u_4 v_4' & v_4 v_4' & v_4' & u_4 & v_4 & 1 \\ u_5 u_5' & v_5 u_5' & u_5' & u_5 v_5' & v_5 v_5' & v_5' & u_5 & v_5 & 1 \\ u_6 u_6' & v_6 u_6' & u_6' & u_6 v_6' & v_6 v_6' & v_6' & u_6 & v_6 & 1 \\ u_7 u_7' & v_7 u_7' & u_7' & u_7 v_7' & v_7 v_7' & v_7' & u_7 & v_7 & 1 \\ u_8 u_8' & v_8 u_8' & u_8' & u_8 v_8' & v_8 v_8' & v_8' & u_8 & v_8 & 1 \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{bmatrix} = 0$$

# Recovering Fundamental Matrix

- 8-point algorithm
  - Construct linear system using corresponding image points

$$W\mathbf{f} = 0$$

How to solve it?

$$\begin{bmatrix} u_1u_1' & v_1u_1' & u_1' & u_1v_1' & v_1v_1' & v_1' & u_1 & v_1 & 1 \\ u_2u_2' & v_2u_2' & u_2' & u_2v_2' & v_2v_2' & v_2' & u_2 & v_2 & 1 \\ u_3u_3' & v_3u_3' & u_3' & u_3v_3' & v_3v_3' & v_3' & u_3 & v_3 & 1 \\ u_4u_4' & v_4u_4' & u_4' & u_4v_4' & v_4v_4' & v_4' & u_4 & v_4 & 1 \\ u_5u_5' & v_5u_5' & u_5' & u_5v_5' & v_5v_5' & v_5' & u_5 & v_5 & 1 \\ u_6u_6' & v_6u_6' & u_6' & u_6v_6' & v_6v_6' & v_6' & u_6 & v_6 & 1 \\ u_7u_7' & v_7u_7' & u_7' & u_7v_7' & v_7v_7' & v_7' & u_7 & v_7 & 1 \\ u_8u_8' & v_8u_8' & u_8' & u_8v_8' & v_8v_8' & v_8' & u_8 & v_8 & 1 \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{bmatrix} = 0$$

# Recovering Fundamental Matrix

- 8-point algorithm
  - Construct linear system using corresponding image points
  - Solve for **f** using SVD

$$W\mathbf{f} = 0$$

$$W = USV^T$$

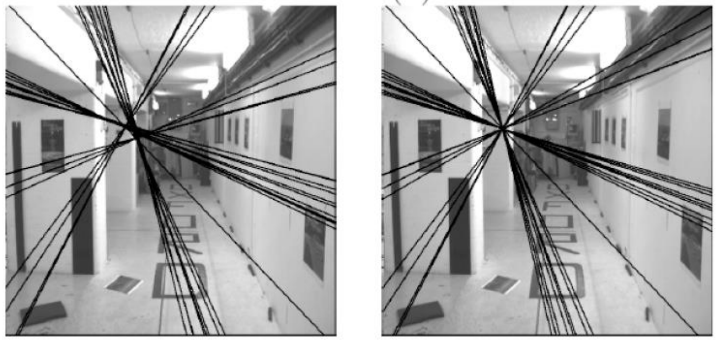
Last column of $V$ gives **f**

$$\begin{bmatrix} u_1u_1' & v_1u_1' & u_1' & u_1v_1' & v_1v_1' & v_1' & u_1 & v_1 & 1 \\ u_2u_2' & v_2u_2' & u_2' & u_2v_2' & v_2v_2' & v_2' & u_2 & v_2 & 1 \\ u_3u_3' & v_3u_3' & u_3' & u_3v_3' & v_3v_3' & v_3' & u_3 & v_3 & 1 \\ u_4u_4' & v_4u_4' & u_4' & u_4v_4' & v_4v_4' & v_4' & u_4 & v_4 & 1 \\ u_5u_5' & v_5u_5' & u_5' & u_5v_5' & v_5v_5' & v_5' & u_5 & v_5 & 1 \\ u_6u_6' & v_6u_6' & u_6' & u_6v_6' & v_6v_6' & v_6' & u_6 & v_6 & 1 \\ u_7u_7' & v_7u_7' & u_7' & u_7v_7' & v_7v_7' & v_7' & u_7 & v_7 & 1 \\ u_8u_8' & v_8u_8' & u_8' & u_8v_8' & v_8v_8' & v_8' & u_8 & v_8 & 1 \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{bmatrix} = 0$$

# Recovering Fundamental Matrix

- 8-point algorithm
  - Construct linear system using corresponding image points
  - Solve for $\mathbf{f}$ using SVD
  - Constraint enforcement (essential step)
    - rank($F$) = 2

Fundamental matrix has rank 2 : $\det(\mathtt{F}) = 0$.



**Left :** Uncorrected $\mathtt{F}$ – epipolar lines are not coincident.

**Right :** Epipolar lines from corrected $\mathtt{F}$.

# Recovering Fundamental Matrix

- 8-point algorithm
  - Construct linear system using corresponding image points
  - Solve for **f** using SVD
  - Constraint enforcement (essential step)
    - rank($F$) = 2

$$\hat{F} = UDV^T \qquad D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \Rightarrow F = U \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T$$

# Recovering Fundamental Matrix

- Problems of 8-point algorithm
  - Sensitive to the origin of coordinates
  - Sensitive to scales


Same scale, different origins (i.e., camera translation)


Image taken using different focal lengths

# Recovering Fundamental Matrix

- Problems of 8-point algorithm
  - Sensitive to the origin of coordinates
  - Sensitive to scales

$$\begin{bmatrix} u_1u'_1 & v_1u'_1 & u'_1 & u_1v'_1 & v_1v'_1 & v'_1 & u_1 & v_1 & 1 \\ u_2u'_2 & v_2u'_2 & u'_2 & u_2v'_2 & v_2v'_2 & v'_2 & u_2 & v_2 & 1 \\ u_3u'_3 & v_3u'_3 & u'_3 & u_3v'_3 & v_3v'_3 & v'_3 & u_3 & v_3 & 1 \\ u_4u'_4 & v_4u'_4 & u'_4 & u_4v'_4 & v_4v'_4 & v'_4 & u_4 & v_4 & 1 \\ u_5u'_5 & v_5u'_5 & u'_5 & u_5v'_5 & v_5v'_5 & v'_5 & u_5 & v_5 & 1 \\ u_6u'_6 & v_6u'_6 & u'_6 & u_6v'_6 & v_6v'_6 & v'_6 & u_6 & v_6 & 1 \\ u_7u'_7 & v_7u'_7 & u'_7 & u_7v'_7 & v_7v'_7 & v'_7 & u_7 & v_7 & 1 \\ u_8u'_8 & v_8u'_8 & u'_8 & u_8v'_8 & v_8v'_8 & v'_8 & u_8 & v_8 & 1 \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{bmatrix} = 0$$

$$\begin{bmatrix} 250906.36 & 183269.57 & 921.81 & 200931.10 & 146766.13 & 738.21 & 272.19 & 198.81 & 1 \\ 2692.28 & 131633.03 & 176.27 & 6196.73 & 302975.59 & 405.71 & 15.27 & 746.79 & 1 \\ 416374.23 & 871684.30 & 935.47 & 408110.89 & 854384.92 & 916.90 & 445.10 & 931.81 & 1 \\ 191183.60 & 171759.40 & 410.27 & 416435.62 & 374125.90 & 893.65 & 465.99 & 418.65 & 1 \\ 48988.86 & 30401.76 & 57.89 & 298604.57 & 185309.58 & 352.87 & 846.22 & 525.15 & 1 \\ 164786.04 & 546559.67 & 813.17 & 1998.37 & 6628.15 & 9.86 & 202.65 & 672.14 & 1 \\ 116407.01 & 2727.75 & 138.89 & 169941.27 & 3982.21 & 202.77 & 838.12 & 19.64 & 1 \\ 135384.58 & 75411.13 & 198.72 & 411350.03 & 229127.78 & 603.79 & 681.28 & 379.48 & 1 \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{bmatrix} = 0$$

Poor numerical conditioning → fix by scaling the data

# Recovering Fundamental Matrix

- ~~8-point algorithm~~
- Normalized 8-point algorithm
  - Idea: normalize image points before constructing the equations
    - Translation: make centroid of image points at origin ← reduce translation effect
    - Scaling: make average distance of points from origin $\sqrt{2}$ ← reduce scaling effect

$$\mathbf{q}_i = T\mathbf{p}_i$$
$$\mathbf{q}_i' = T'\mathbf{q}_i'$$

# Recovering Fundamental Matrix

- ~~8-point algorithm~~
- Normalized 8-point algorithm
  - Normalization of image points (essential step)
  - Solve for $F_q$ using the original 8-point algorithm
    - $F_q$ is the fundamental matrix computed form the normalized image points
    - Same procedure as in original 8-point algorithm

# Recovering Fundamental Matrix

- ~~8 point algorithm~~
- Normalized 8-point algorithm
  - Normalization of image points (essential step)
  - Solve for $F_q$ using the original 8-point algorithm
  - De-normalization (essential step)

$$\mathbf{q}'^T F_q \mathbf{q} = 0$$

Normalized image points

$$\mathbf{q} = T\mathbf{p} \qquad \mathbf{q}' = T'\mathbf{p}'$$

Original image points

$$\Rightarrow (T'\mathbf{p}')^T F_q (T\mathbf{p}) = 0 \Rightarrow \mathbf{p}'^T \underbrace{(T'^T F_q T)}_{F} \mathbf{p} = 0$$

$F_q$: fundamental matrix computed form normalized image points

$F$ : the expected fundamental matrix

# Next lecture

- 2-view 3D reconstruction
  - Camera calibration
  - Triangulation
- Structure from Motion
  - Go beyond two views
  - Simultaneously
    - recover 3D structure
    - Refine camera parameters