Lecture 5
Reconstruct 3D Geometry

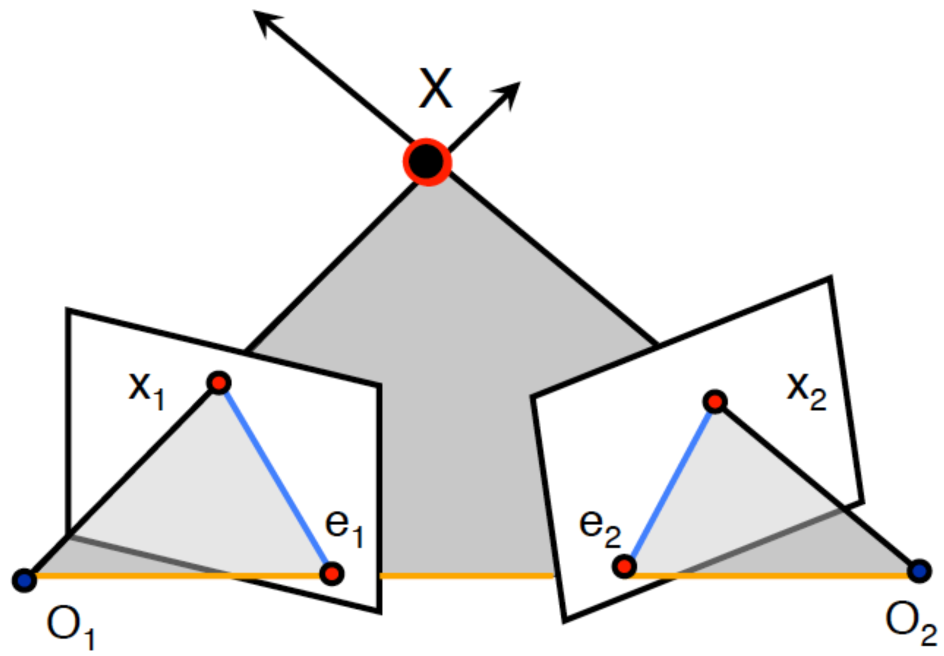
Liangliang Nan

Today's Agenda

- Review of Epipolar Geometry
- Image Matching (self study)
- Reconstruct 3D Geometry
  - 3D from 2 views
    - Estimate fundamental matrix
    - Recover relative pose
    - Triangulation
  - 3D from more views
    - Structure from motion
      - Bundle adjustment

Review of Epipolar Geometry

- Epipolar Geometry
  - Baseline
    - The line between the two camera centers
  - Epipolar plane
    - The plane defined by X, O_1, and O_2
  - Epipoles
    - ∩ of baseline and image plane
    - Projection of the other camera center
  - Epipolar lines
    - ∩ of epipolar plane with the image plane



Review of Epipolar Geometry

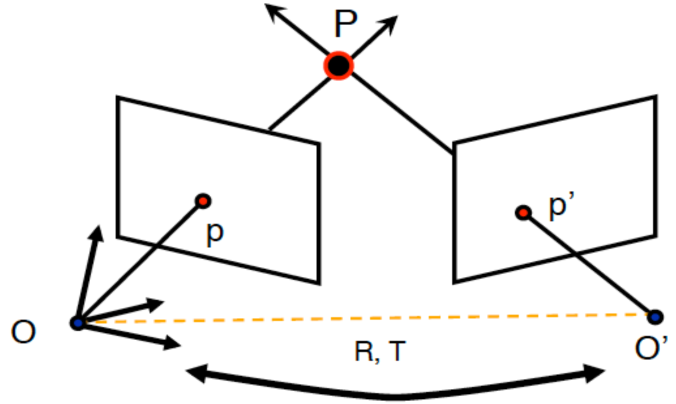
- Essential matrix
  - Canonical camera assumption

$$p'^T E\, p = 0,\ E = [T_X]R \qquad K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Fundamental matrix (most important concept in 3DV)

$$p'^T F\, p = 0,\ F = K'^{-T}[T_\times]RK^{-1}$$

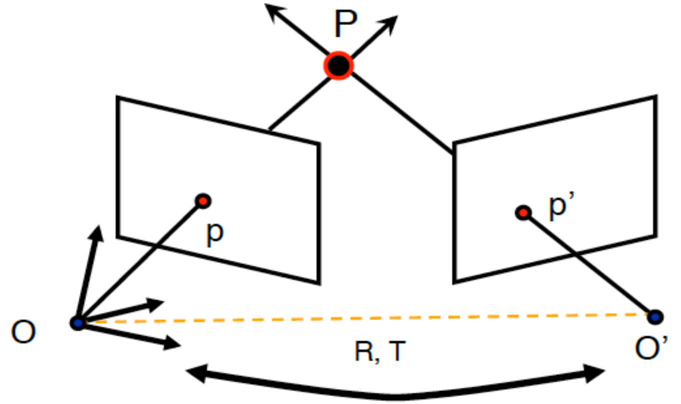
  - Relate matching points of different views
    - No need 3D point location
    - No need intrinsic parameters
    - No need extrinsic parameters



Review of Epipolar Geometry

- Fundamental matrix
  - 3 by 3
  - homogeneous matrix
  - 7 degrees of freedom
    - 9 elements
    - scale ambiguity (scale doesn't matter)
    - determinant(F) = 0

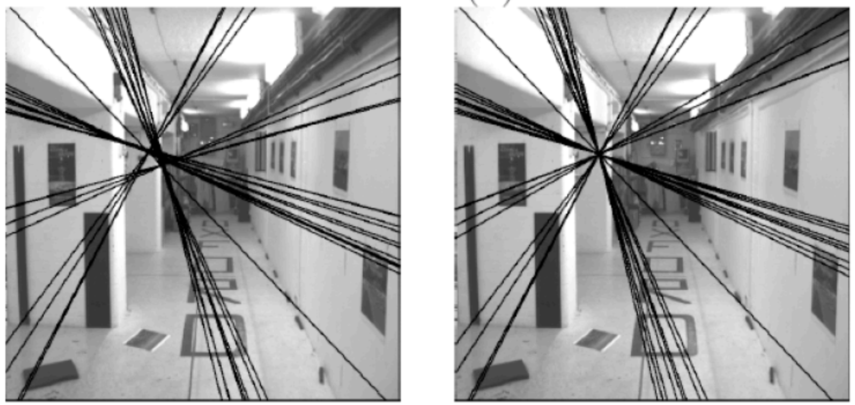
$$p'^T F\, p = 0, \qquad F = K'^{-T}[T_\times]RK^{-1}$$



Review of Epipolar Geometry

- Fundamental matrix
  - 3 by 3
  - homogeneous matrix
  - 7 degrees of freedom
    - 9 elements
    - scale ambiguity (scale doesn’t mat
    - determinant(F) = 0
  - rank(F) = 2

Fundamental matrix has rank 2 : $\det(\mathtt{F}) = 0$.



Left : Uncorrected $\mathtt{F}$ – epipolar lines are not coincident.

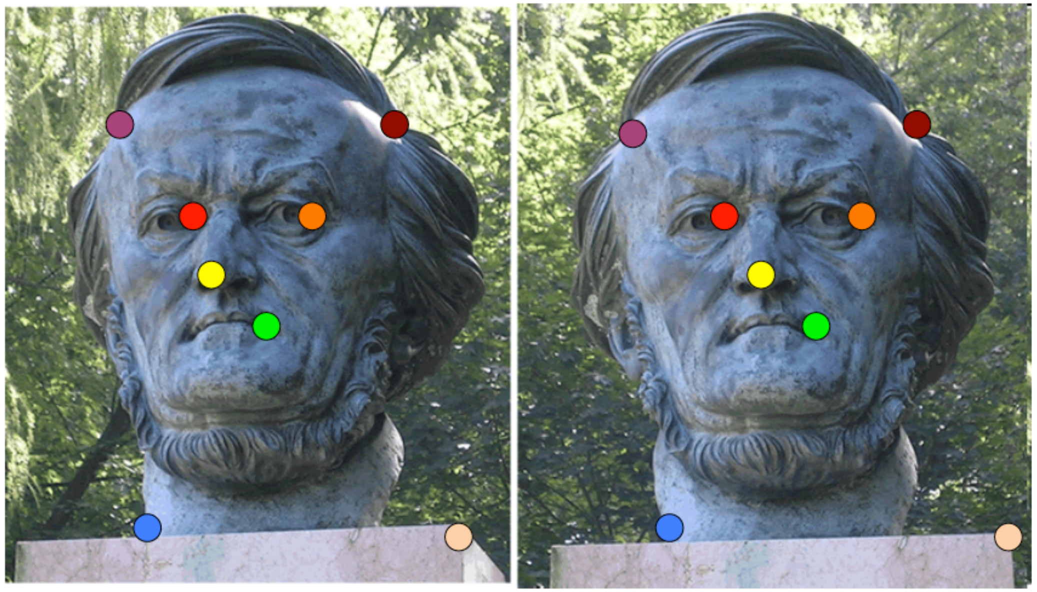
Right : Epipolar lines from corrected $\mathtt{F}$.

Review of Epipolar Geometry

- Recover F from corresponding image points
  - 8 unknown parameters
  - Each point pair gives a single linear constraint

$$\begin{cases} p_i = (u_i, v_i, 1) \\ p'_i = (u'_i, v'_i, 1) \end{cases} + \quad p'^T F\, p = 0,$$

$$\begin{bmatrix} u_i u'_i & v_i u'_i & u'_i & u_i v'_i & v_i v'_i & v'_i & u_i & v_i & 1 \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{bmatrix} = 0$$



Review of Epipolar Geometry

- Recover F from corresponding image points
  - 8 unknown parameters
  - Each point pair gives a single linear constraint
  - 8-point algorithm (>= 8 pairs)
  - 7-point algorithm does exist but less popular

$$\begin{bmatrix} u_1u_1' & v_1u_1' & u_1' & u_1v_1' & v_1v_1' & v_1' & u_1 & v_1 & 1 \\ u_2u_2' & v_2u_2' & u_2' & u_2v_2' & v_2v_2' & v_2' & u_2 & v_2 & 1 \\ u_3u_3' & v_3u_3' & u_3' & u_3v_3' & v_3v_3' & v_3' & u_3 & v_3 & 1 \\ u_4u_4' & v_4u_4' & u_4' & u_4v_4' & v_4v_4' & v_4' & u_4 & v_4 & 1 \\ u_5u_5' & v_5u_5' & u_5' & u_5v_5' & v_5v_5' & v_5' & u_5 & v_5 & 1 \\ u_6u_6' & v_6u_6' & u_6' & u_6v_6' & v_6v_6' & v_6' & u_6 & v_6 & 1 \\ u_7u_7' & v_7u_7' & u_7' & u_7v_7' & v_7v_7' & v_7' & u_7 & v_7 & 1 \\ u_8u_8' & v_8u_8' & u_8' & u_8v_8' & v_8v_8' & v_8' & u_8 & v_8 & 1 \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{bmatrix} = 0$$

$$W\mathbf{f} = 0$$

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      - Bundle adjustment

Image Matching

- SIFT, Surf …
- RANSAC

Image Matching

- SIFT

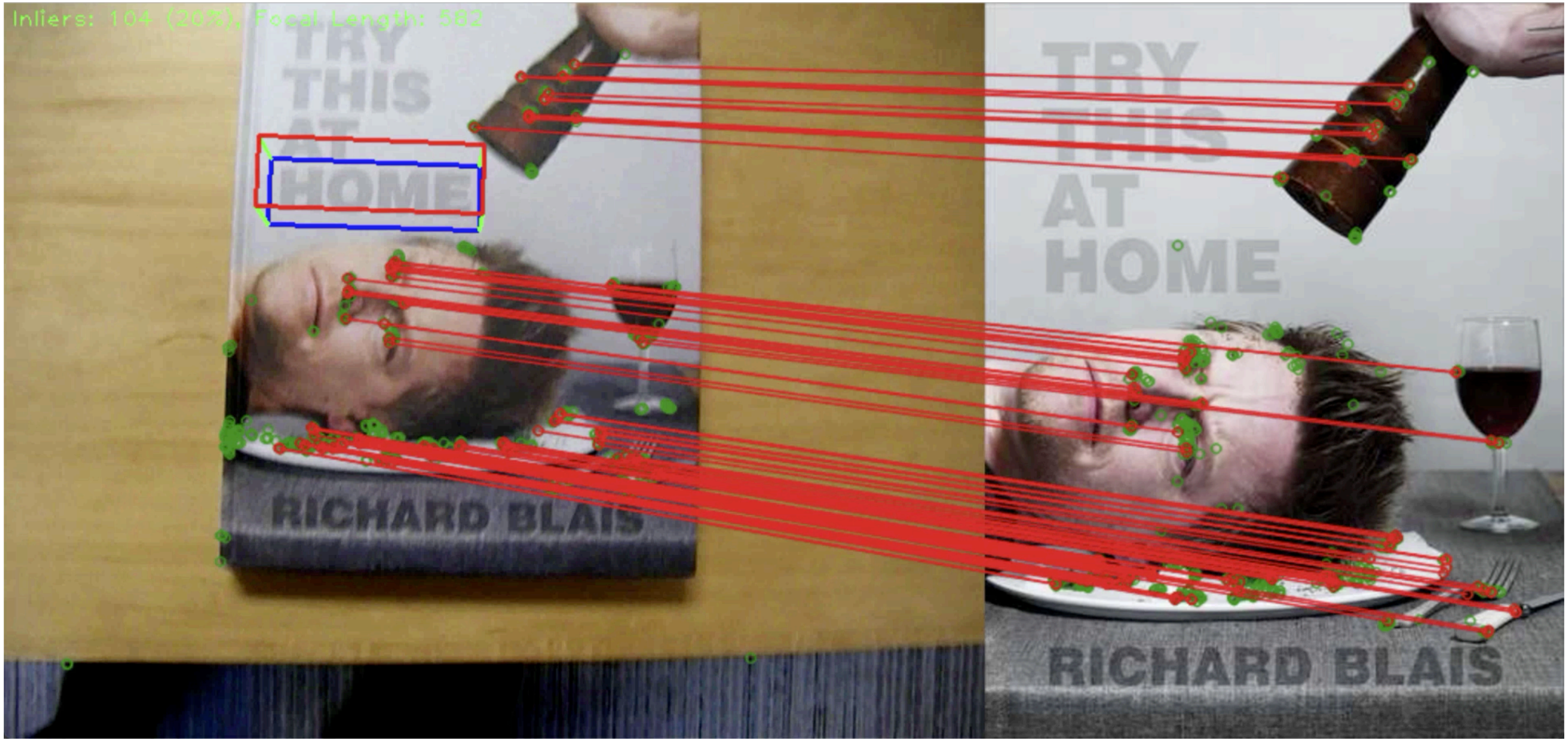
David Lowe

Computer Science Dept., University of British Columbia
Verified email at cs.ubc.ca - Homepage
Computer Vision   Object Recognition

FOLLOW

TITLE	CITED BY	YEAR
Distinctive image features from scale-invariant keypoints DG Lowe International journal of computer vision 60 (2), 91-110	61454	2004

Image Matching



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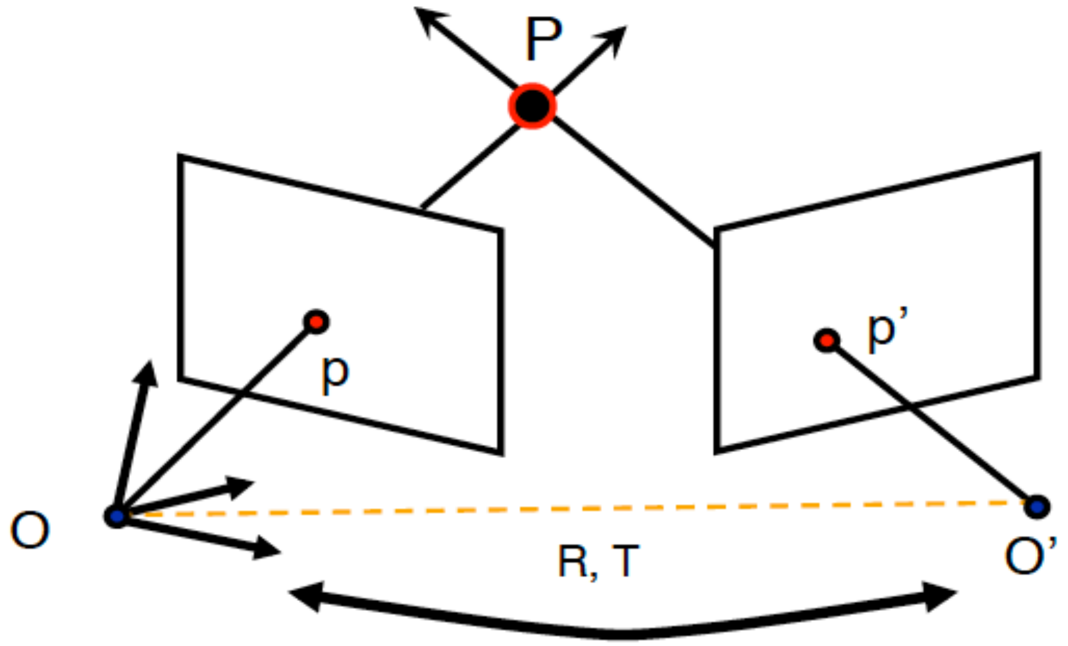
3D from 2 Views

- The general idea

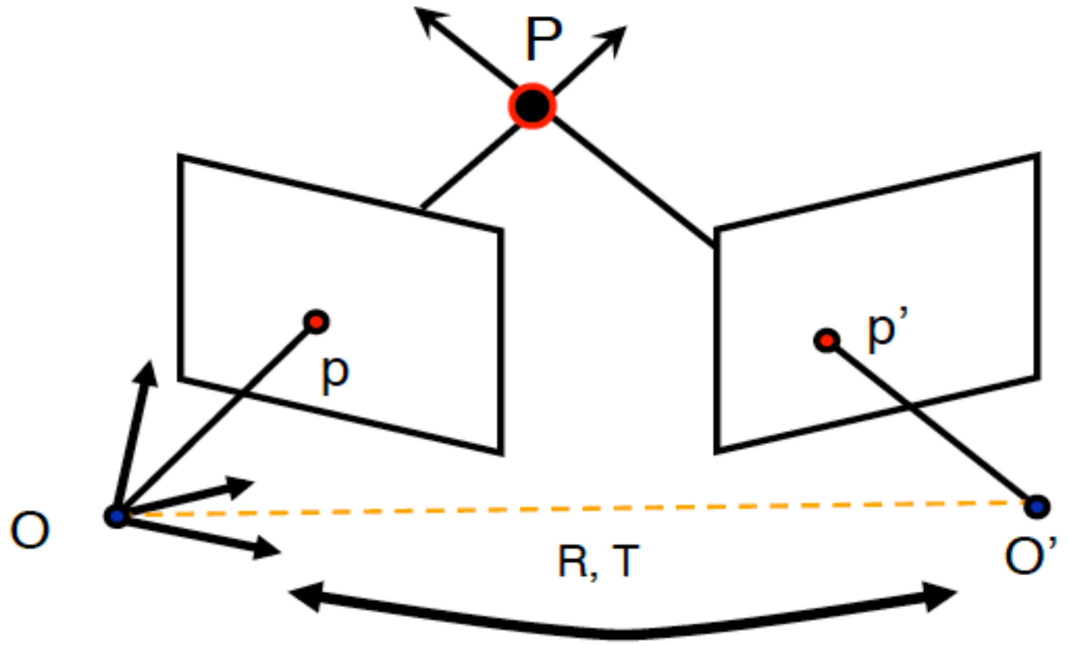


3D from 2 Views

- What information is needed?
  - Corresponding image points
    - Image matching techniques
  - Intrinsic camera parameters
    - Camera calibration
  - Extrinsic camera parameters
    - Recover from image points?

$$p'^{T} F \, p = 0,$$

$$F = K'^{-T} [T_{\times}] R K^{-1}$$



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      - Bundle adjustment
- Image Matching

Recover F from Matched Image Points

- 8-point algorithm (>= 8 point pairs)
  - Sensitive to noise
  - Sensitive to origin of coordinates
  - Sensitive to scales

Same scale, different origins

Image taken using different focal lengths

Recover F from Matched Image Points

- 8-point algorithm (>= 8 point pairs)
- Normalized 8-point algorithm
  - Idea: normalize before constructing the equations
    - Translation: the centroid of the image points is at origin
    - Scaling: average distance of points from origin is $\sqrt{2}$

$$q_i = T p_i \qquad q_i' = T' p_i'$$

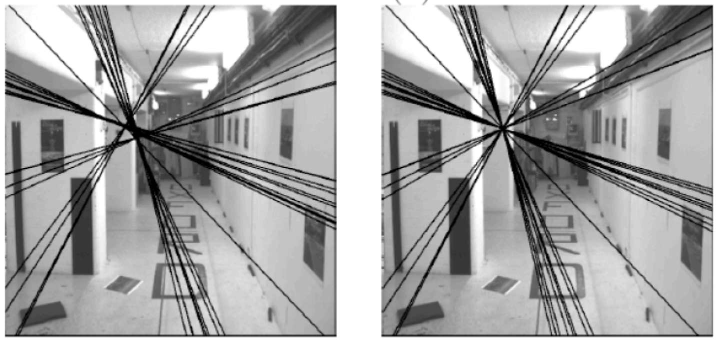
Recover F from Matched Image Points

- 8-point algorithm (>= 8 point pairs)
- Normalized 8-point algorithm
  - – Idea: normalize before constructing the equations
  - – Construct linear system using the normalized points
    - Same as in the original 8-point algorithm
  - – Solve using SVD
    - Same as in the original 8-point algorithm

Recover F from Matched Image Points

- 8-point algorithm (>= 8 point pairs)
- Normalized 8-point algorithm
  - Idea: normalize before constructing the equations
  - Construct linear system using the normalized points
  - Solve using SVD
  - Constraint enforcement
    - rank(F) = 2

Fundamental matrix has rank 2 : $\det(\mathbf{F}) = 0$.



Left : Uncorrected F – epipolar lines are not coincident.

Right : Epipolar lines from corrected F.

Recover F from Matched Image Points

- 8-point algorithm (>= 8 point pairs)
- Normalized 8-point algorithm
  - Idea: normalize before constructing the equations
  - Construct linear system using the normalized points
  - Solve using SVD
  - Constraint enforcement
    - rank(F) = 2

$$\hat{F} = U\Sigma V^T \qquad F = U \begin{bmatrix} \Sigma_1 & 0 & 0 \\ 0 & \Sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T$$

Recover F from Matched Image Points

- 8-point algorithm (>= 8 point pairs)
- Normalized 8-point algorithm
  - Idea: normalize before constructing the equations
  - Construct linear system using the normalized points
  - Solve using SVD
  - Constraint enforcement
  - De-normalization
    - Apply the **inverse** of the transformation

$$q_i = T p_i \qquad q'_i = T' p'_i$$

$$F = T'^T F_q T$$

See handout on “Epipolar geometry”

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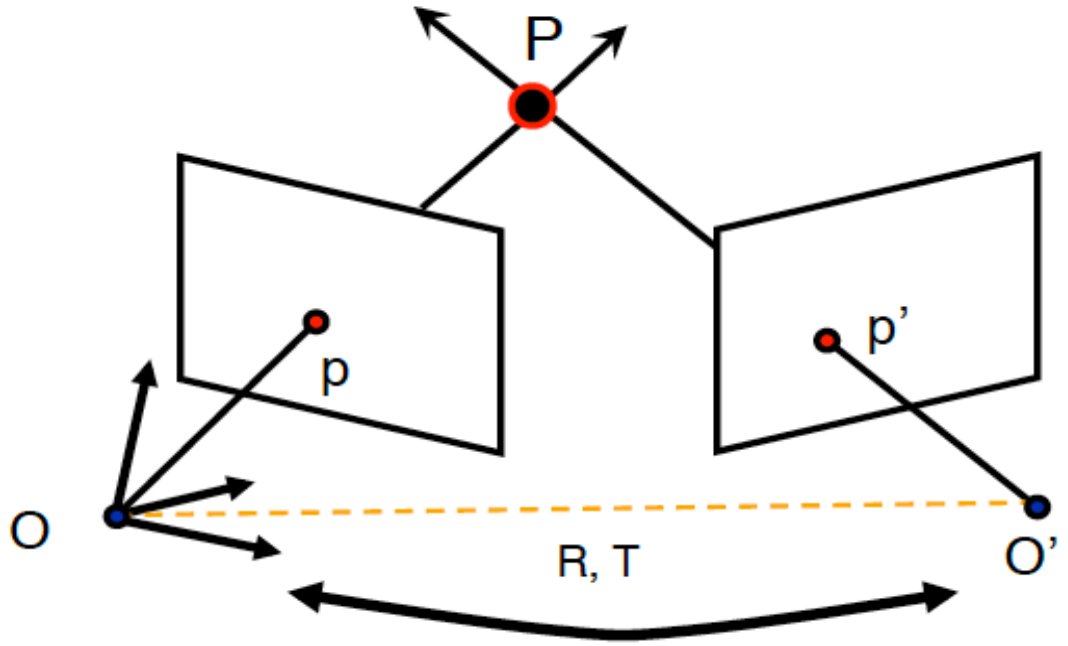
Relative Pose from Fundamental Matrix

- Essential matrix from fundamental matrix
  - Known intrinsic parameters
    - Estimation
    - Calibration

$$F = K'^{-T}[t_{\times}]RK^{-1}$$

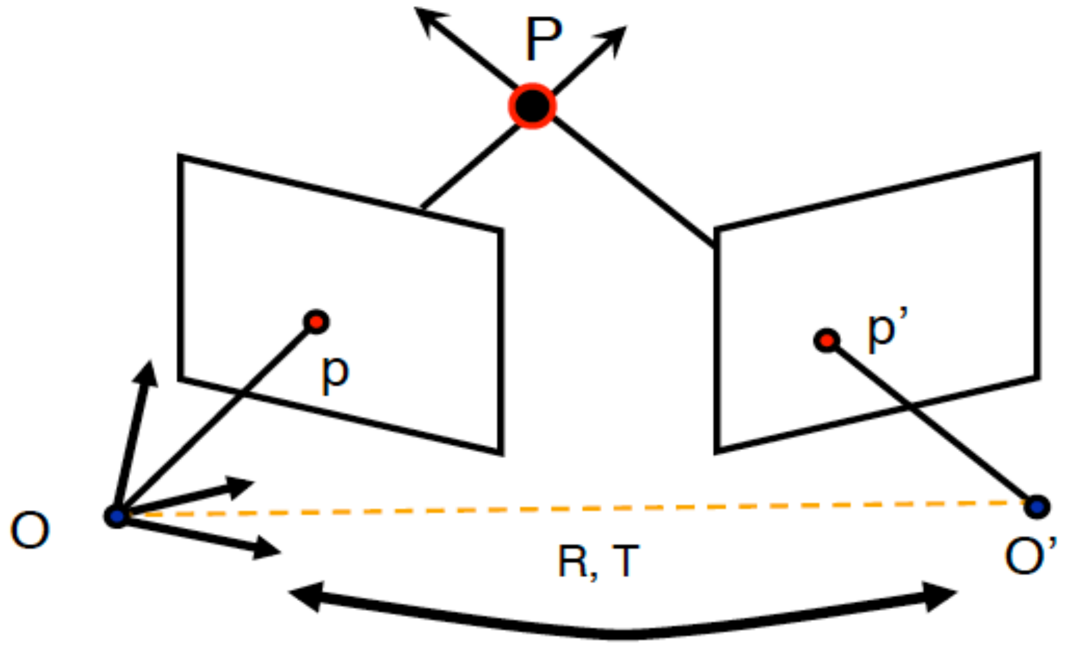
$$E = [t_{\times}]R = K'^{T}FK$$



Relative Pose from Fundamental Matrix

- Essential matrix from fundamental matrix
- Relative pose from essential matrix

$$E = [t_{\times}]R$$



Relative Pose from Fundamental Matrix

- Essential matrix from fundamental matrix
- Relative pose from essential matrix
  - SVD of E

$$W = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad E = U\Sigma V^T$$

Relative Pose from Fundamental Matrix

- Essential matrix from fundamental matrix
- Relative pose from essential matrix
  - SVD of E
  - determinant(R) > 0
    - Two potential values

$$W = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad E = U\Sigma V^T$$

$$R = (\det UWV^T)UWV^T \text{ or } (\det UW^TV^T)UW^TV^T$$

Relative Pose from Fundamental Matrix

- Essential matrix from fundamental matrix
- Relative pose from essential matrix
  - SVD of E
  - determinant(R) > 0
    - Two potential values
  - T up to a sign
    - Two potential values

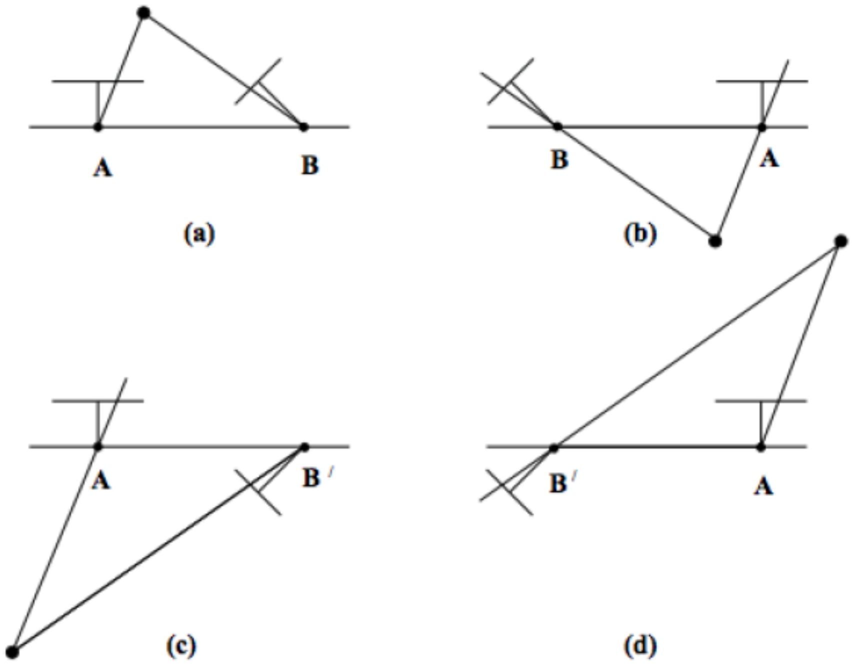
$$W = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad E = U \Sigma V^T$$

$$R = (\det UWV^T)UWV^T \text{ or } (\det UW^TV^T)UW^TV^T$$

$$t = \pm U \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \pm u_3$$

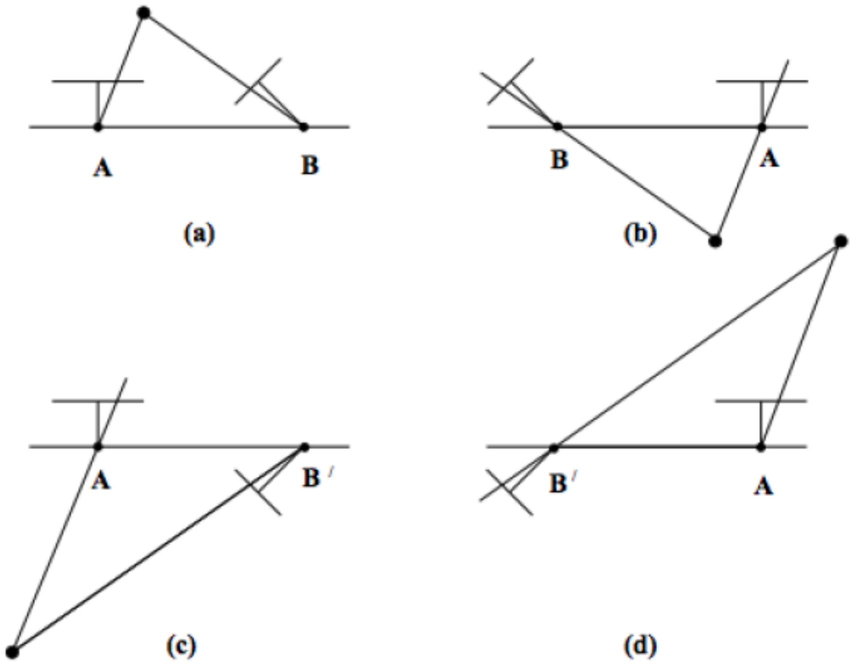
Relative Pose from Fundamental Matrix

- Essential matrix from fundamental matrix
- Relative pose from essential matrix
  - R: two potential values
  - T: two potential values



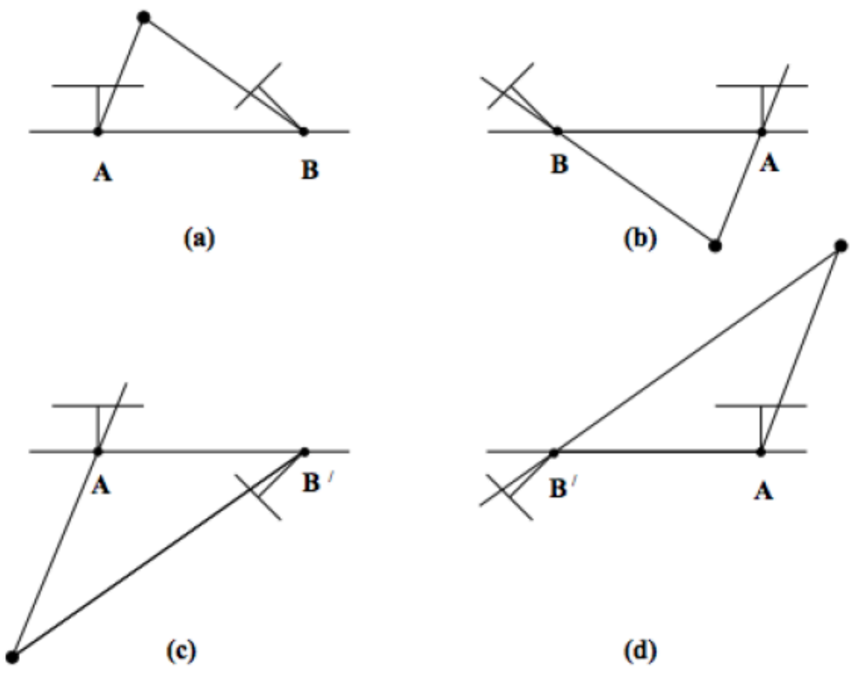
Relative Pose from Fundamental Matrix

- Essential matrix from fundamental matrix
- Relative pose from essential matrix
  - R: two potential values
  - T: two potential values
  - 3D points must be in front of both cameras



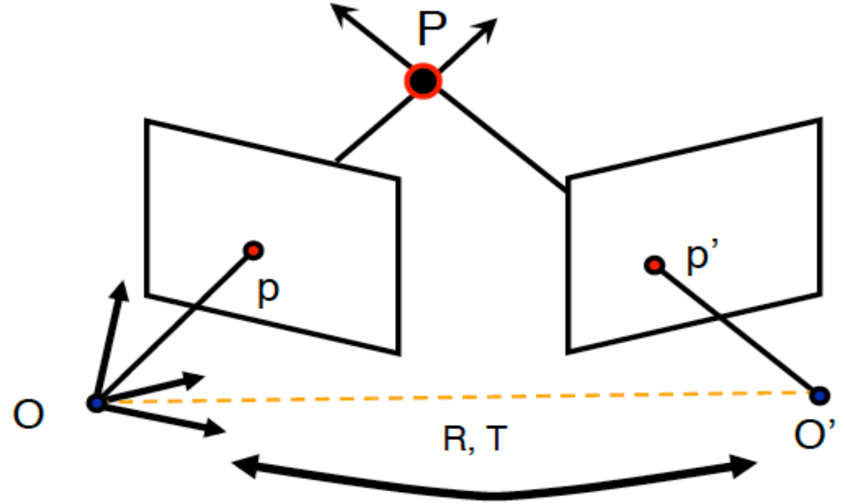
Relative Pose from Fundamental Matrix

- Essential matrix from fundamental matrix
- Relative pose from essential matrix
  - R: two potential values
  - T: two potential values
  - 3D points must be in front of both cameras
    - Reconstruct 3D points
      - using all potential pairs of R and t
    - Count the number of points in front of cameras
    - The pair giving max front points is correct



Relative Pose from Fundamental Matrix

- Essential matrix from fundamental matrix
- Relative pose from essential matrix
  - R: two potential values
  - T: two potential values
  - 3D points must be in front of both cameras
    - First camera
      - P.z > 0 ?
    - Second camera
      - P in 2nd camera's coordinate system: Q = R * P + t
      - Q.z > 0 ?



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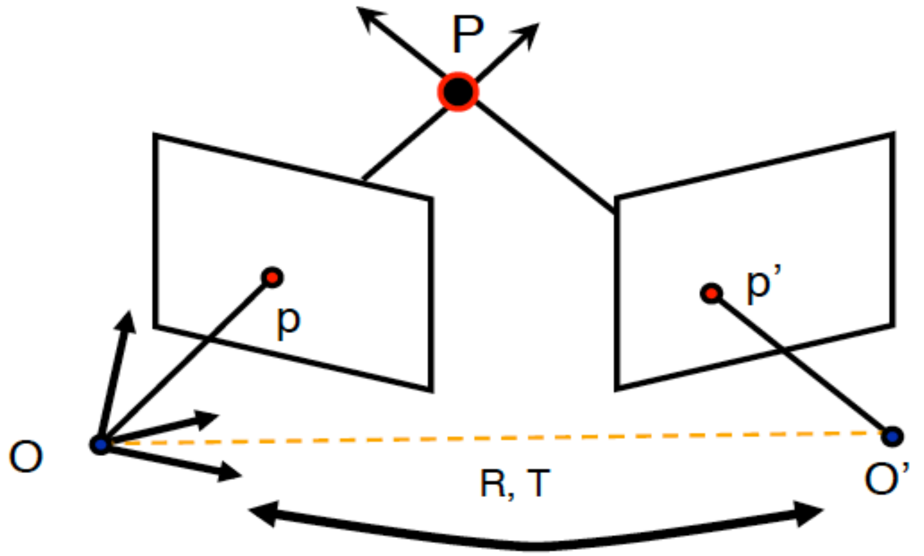
    - Structure from motion
      - Bundle adjustment

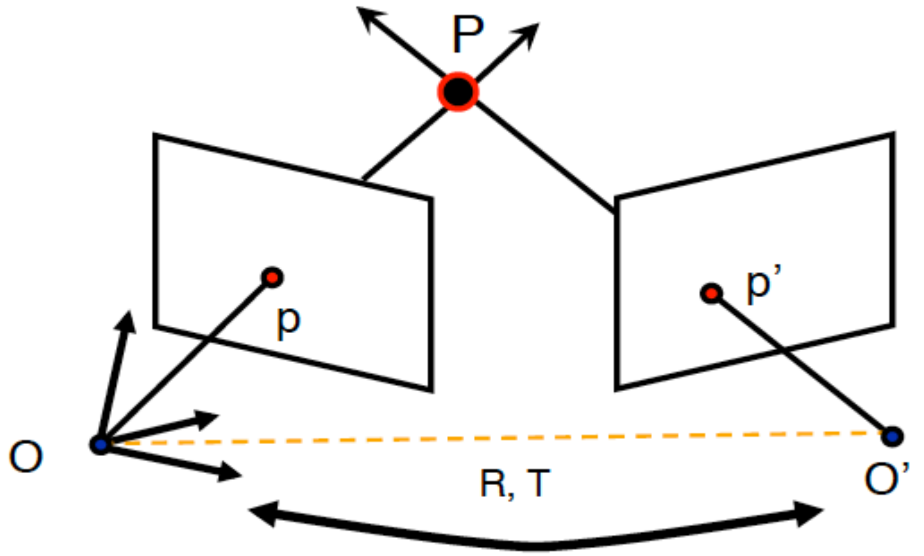
Triangulation

- Find coordinates of 3D point from its projection into two views
  - Known camera intrinsic parameters (K)
  - Known relative orientation (R) and offset (T)



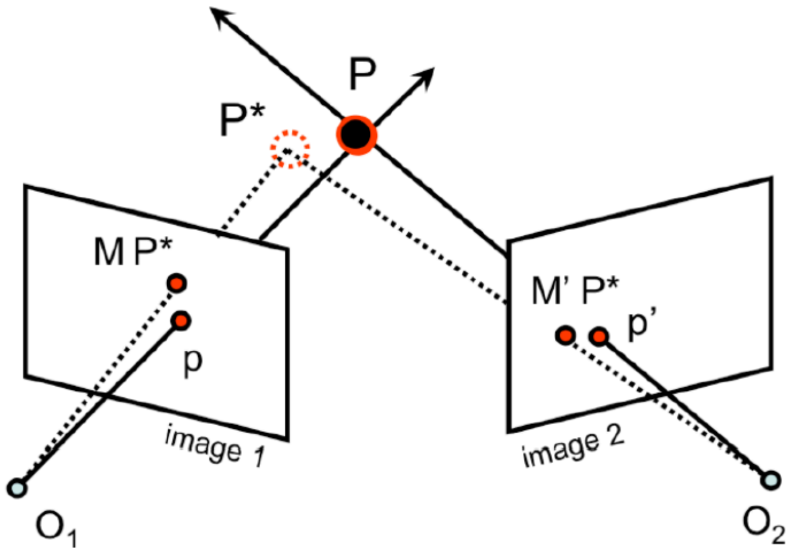
Triangulation

- Find coordinates of 3D point from its projection into two views
  - Known camera intrinsic parameters (K)
  - Known relative orientation (R) and offset (T)
  - In theory, P is ∩ of the two lines of sight



Triangulation

- Find coordinates of 3D point from its projection into two views
  - Known camera intrinsic parameters (K)
  - Known relative orientation (R) and offset (T)
  - In theory, P is ∩ of the two lines of sight
    - Straightforward and mathematically sound
    - Do not work well
      - Noisy in observation
      - K, R, T are not precise



Triangulation

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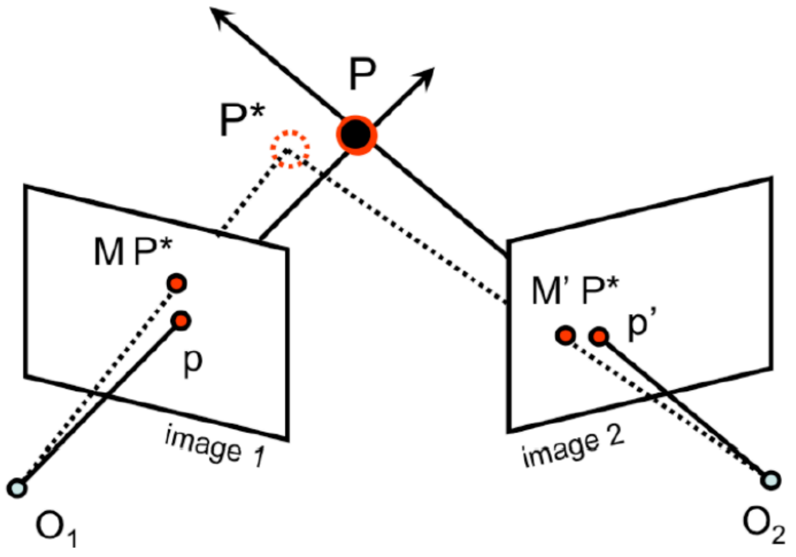
    - Do not work well
      - Noisy in observation
      - K, R, T are not precise
  - A linear method for triangulation
  - A non- linear method for triangulation



A Linear Method for Triangulation

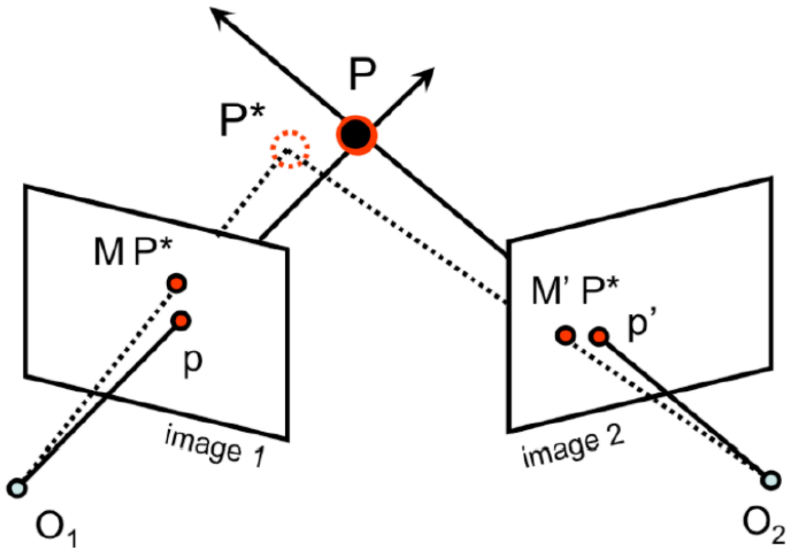
Two image points

$$p = MP = (x, y, 1)$$

$$p' = M'P = (x', y', 1)$$

By the definition of the cross product

$$p \times (MP) = 0$$



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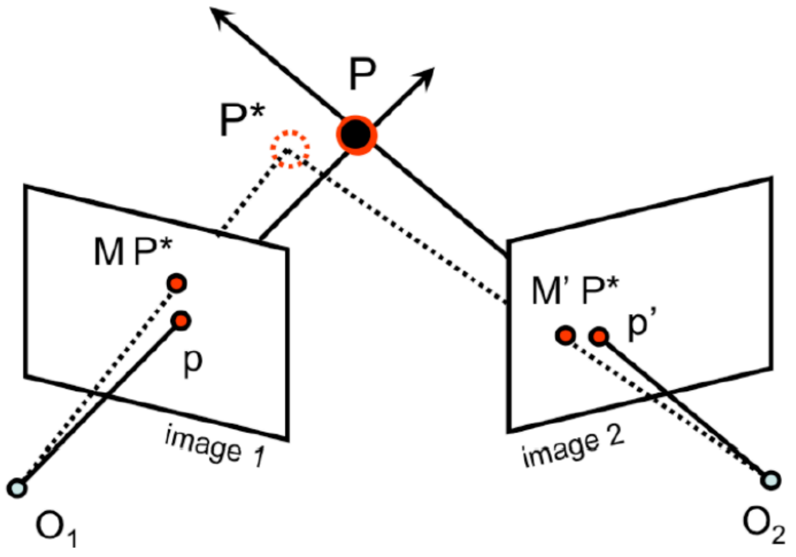
By the definition of the cross product

$$p \times (MP) = 0$$

Similar constraints can also be formulated for p' and M'.

$$x(M_3P) - (M_1P) = 0$$
$$y(M_3P) - (M_2P) = 0$$
$$x(M_2P) - y(M_1P) = 0$$



A Linear Method for Triangulation

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$$x(M_3P) - (M_1P) = 0$$
$$y(M_3P) - (M_2P) = 0$$
$$x(M_2P) - y(M_1P) = 0$$

$$A = \begin{bmatrix} xM_3 - M_1 \\ yM_3 - M_2 \\ x'M_3' - M_1' \\ y'M_3' - M_2' \end{bmatrix}$$

$$AP = 0$$

A Linear Method for Triangulation

- Advantages
  - Easy to solve and very efficient
  - Any number of corresponding image points
  - Can handle multiple views
  - Used as initialization to advanced methods

$$AP = 0$$

$$A = \begin{bmatrix} xM_3 - M_1 \\ yM_3 - M_2 \\ x'M_3' - M_1' \\ y'M_3' - M_2' \end{bmatrix}$$

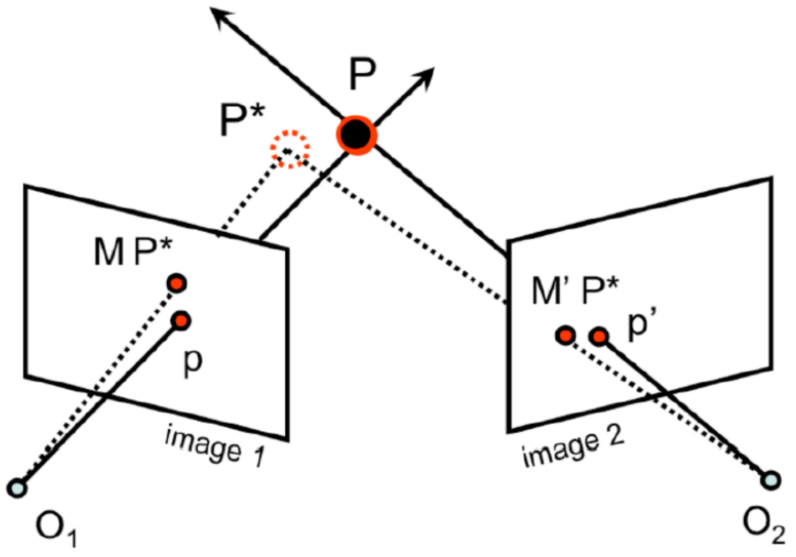
The Non-linear Method for Triangulation

- Minimize the reprojection error

$$\min_{\hat{P}} \sum_i \|M\hat{P}_i - p_i\|^2 + \|M'\hat{P}_i - p_i'\|^2$$

Reprojection error

  - Gauss-Newton's method
  - Levenberg-Marquardt



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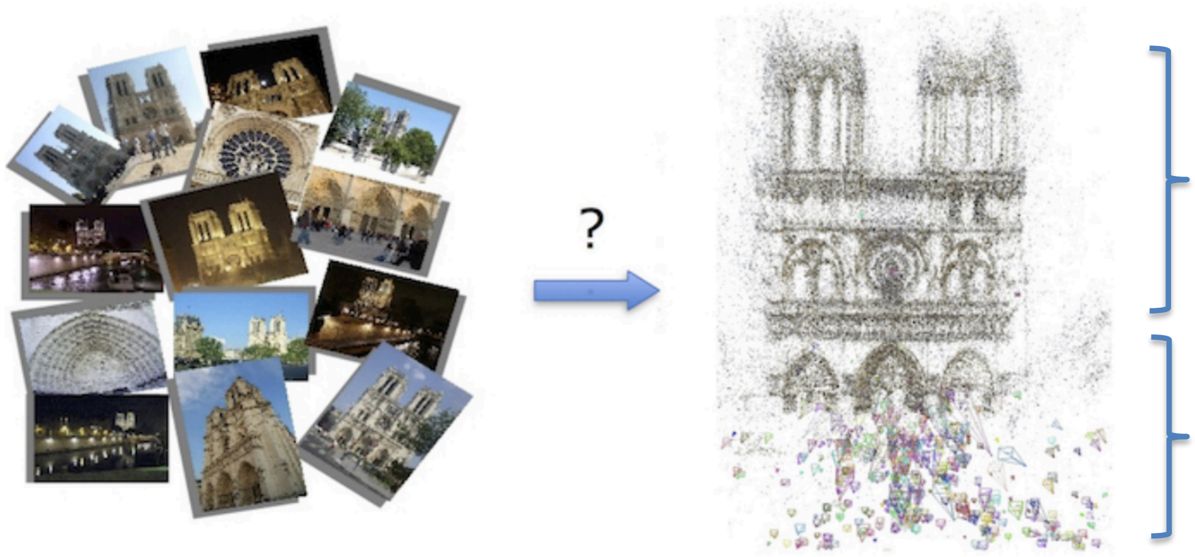
Structure from Motion

- Structure?
  - 3D geometry of the scene/object
- Motion?
  - Camera locations and orientations



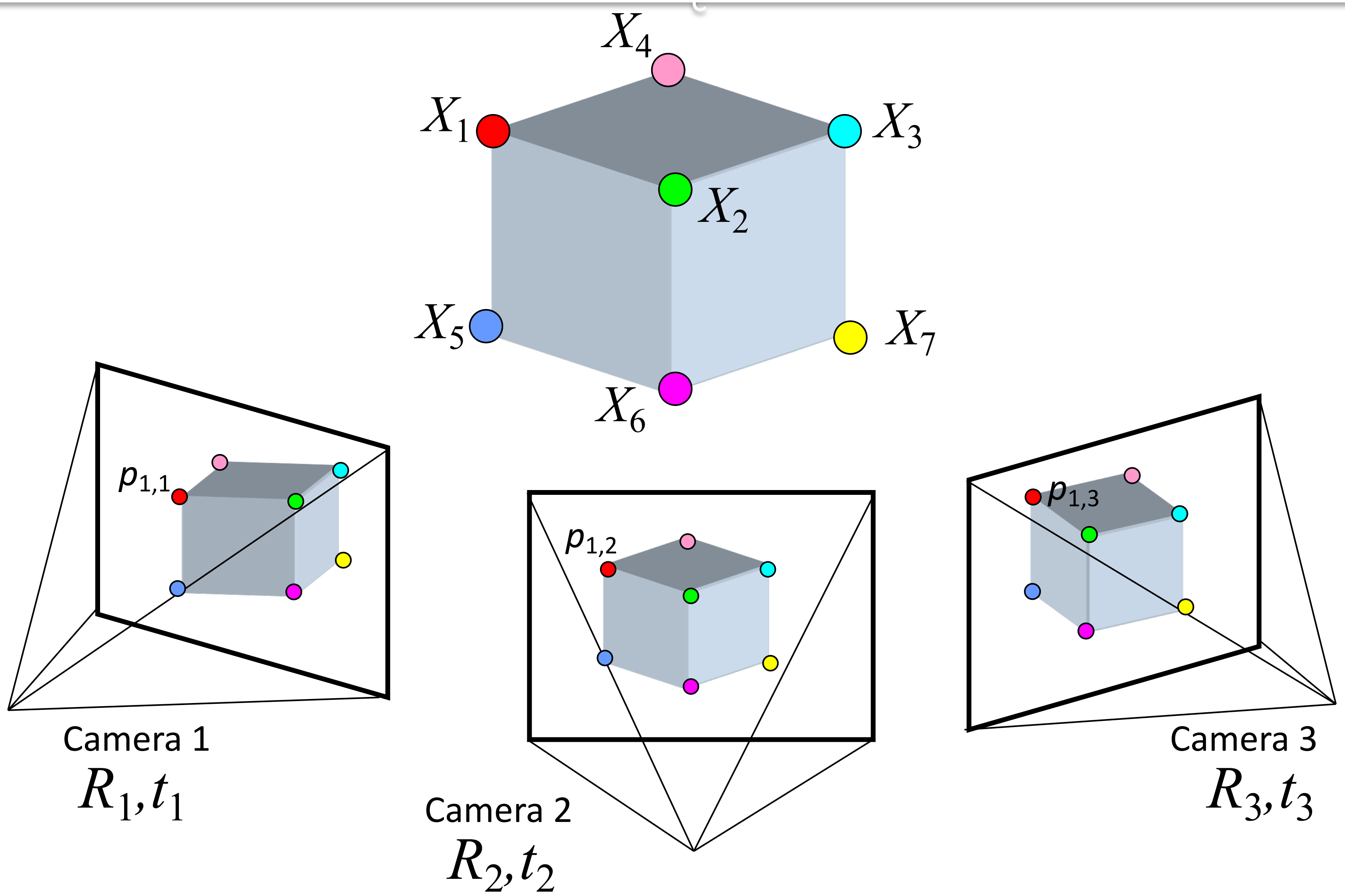
3D structure

Camera motion

Structure from Motion

- Structure
  - 3D geometry of the scene/object
- Motion
  - Camera locations and orientations
- Structure from Motion
  - Compute the geometry from moving cameras?
  - Simultaneously recovering structure and motion

Structure from Motion



Bundle Adjustment

- Minimize sum of squared re-projection errors:

$$g(\mathbf{X},\mathbf{R},\mathbf{T}) = \sum_{i=1}^{m}\sum_{j=1}^{n} \underbrace{w_{ij}}_{\text{indicator variable: is point } i \text{ visible in image } j\text{ ?}} \cdot \left\| \underbrace{\mathbf{P}(\mathbf{x}_i,\mathbf{R}_j,\mathbf{t}_j)}_{\text{predicted image points}} - \underbrace{\begin{bmatrix} u_{i,j} \\ v_{i,j} \end{bmatrix}}_{\text{observed image points}} \right\|^2$$

predicted image points

observed image points

indicator variable: is point *i* visible in image *j* ?

Bundle Adjustment

- Minimize sum of squared re-projection errors:

$$g(\mathbf{X}, \mathbf{R}, \mathbf{T}) = \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} \cdot \left\| \mathbf{P}(\mathbf{x}_i, \mathbf{R}_j, \mathbf{t}_j) - \begin{bmatrix} u_{i,j} \\ v_{i,j} \end{bmatrix} \right\|^2$$

- Minimizing this function is called *bundle adjustment*
  - Optimized using non-linear least squares,
    e.g. Levenberg-Marquardt

Bundle Adjustment

- Minimize sum of squared re-projection errors:

$$g(\mathbf{X}, \mathbf{R}, \mathbf{T}) = \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} \cdot \left\| \mathbf{P}(\mathbf{x}_i, \mathbf{R}_j, \mathbf{t}_j) - \begin{bmatrix} u_{i,j} \\ v_{i,j} \end{bmatrix} \right\|^2$$

- Minimizing this function is called *bundle adjustment*
- Initialization
  - – From chained 2-view reconstruction
    - Relative motion can be estimated from the corresponding images points
    - 3D points can be estimated from the relative motion using triangulation
  - – Global optimization techniques when the poses and the 3D structure are initialized arbitrarily.

Bundle Adjustment

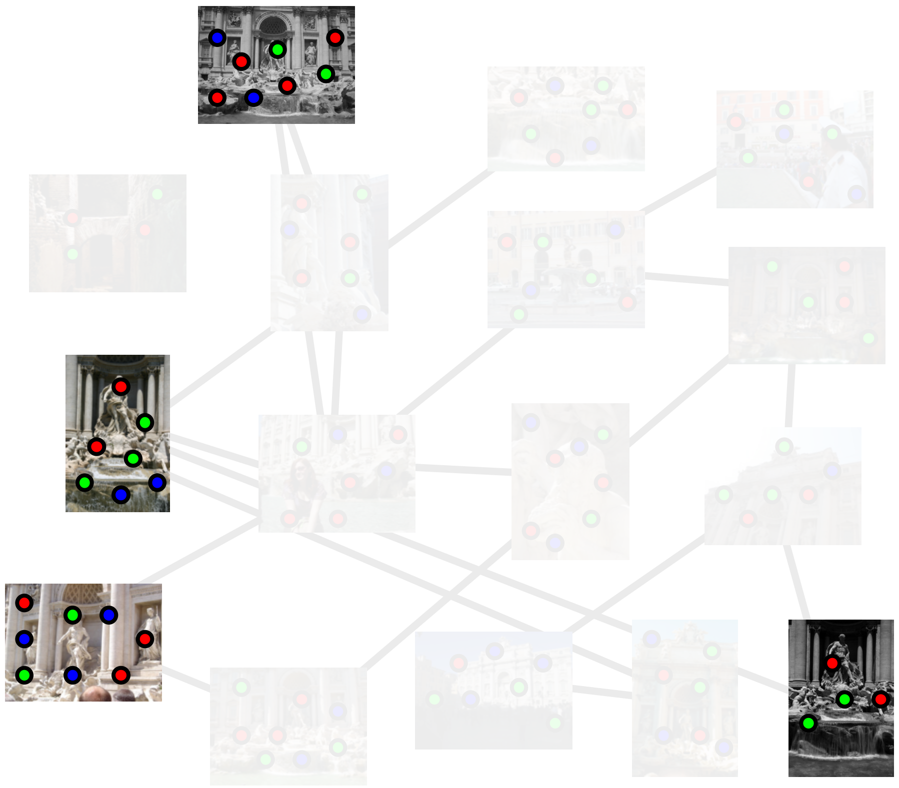
- What are the variables?
  - Camera intrinsic parameters, extrinsic parameters
  - Coordinates of the 3D points
- How many variables per camera?
- How many variables per point?

500 input photos

100,000 3D points

= Very large optimization problem

Incremental SfM

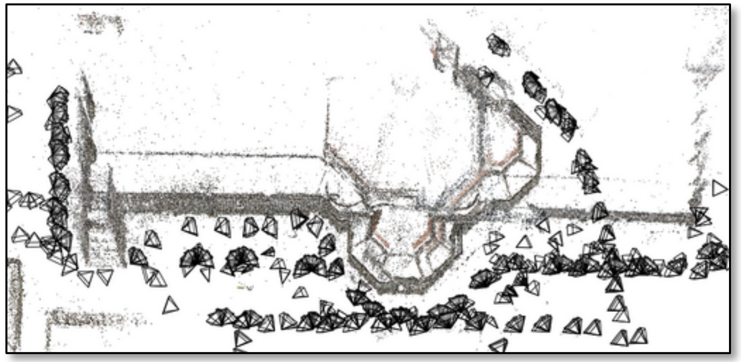


Structure from Motion

Failure Cases

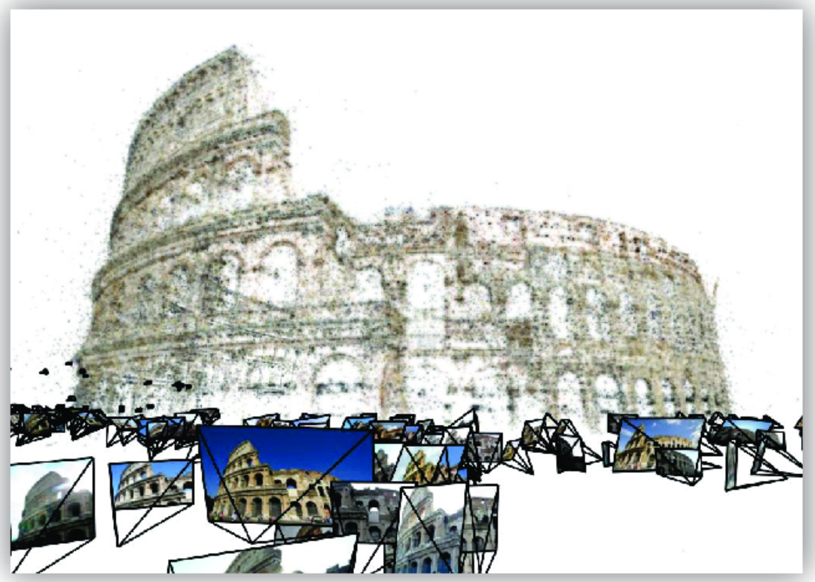
- Repetitive structures






Next Lecture

- Multi-view Stereo
  - Obtaining dense point clouds



Images + camera information

Dense 3d point cloud