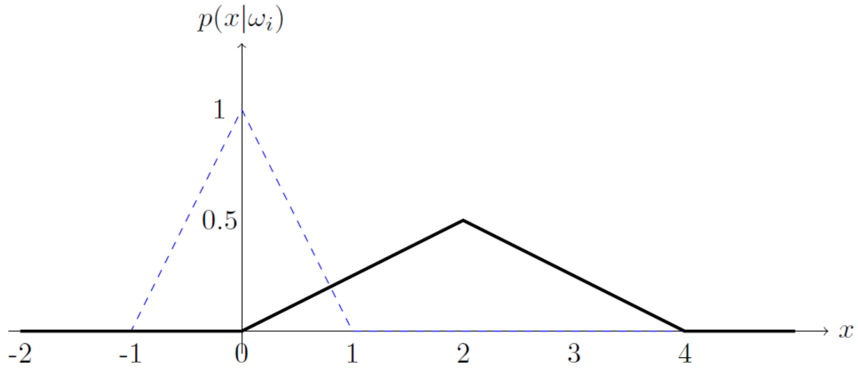

Figure 1.1: The class-conditional probabilities of two classes $p(x|\omega_1)$ (dashed blue line) and $p(x|\omega_2)$ (solid black line) in a 1-dimensional feature space.

1. Two class conditional probability density functions are given in the figure above. The first class ω_1 is represented by a dashed blue line and the second class ω_1 is represented with a solid black line. Two classes have equal prior probability:

$$P(\omega_1) = P(\omega_2) = 0.5$$

(a) Use the Bayes' rule to derive the class posterior probabilities of the following objects:

- x = -0.5;
- x = +0.5;
- x = 3;

To which class are the objects therefore assigned?

(b) What is the decision boundary of the Bayes classifier?

2. Revisit the question 1, assume the prior probabilities of two classes have changed:

$$P(\omega_1) = \frac{1}{3}$$

$$P(\omega_2) = \frac{2}{3}$$

Again, what is the decision boundary of the Bayes classifier?

Answer:
1. (a) Applying Bayes rule,

$$P(\omega_i|x) = \frac{P(\omega_i)\,P(x|\omega_i)}{P(x)} = \frac{P(\omega_i)\,P(x|\omega_i)}{\sum_{i=1,2} P(\omega_i)\,P(x|\omega_i)} = \frac{P(x|\omega_i)}{\sum_{i=1,2} P(x|\omega_i)}$$

We can easily get that:
x=-0.5, $P(\omega_1|x) = 1$, $P(\omega_2|x) = 0$, this object belongs to ω_1;
x=3, $P(\omega_1|x) = 0$, $P(\omega_2|x) = 1$, this object belongs to ω_2;

When x=+0.5, $P(x|\omega_1) = \frac{1}{2}, P(x|\omega_2) = \frac{1}{8}$. Therefore,
$P(\omega_1|x) = \frac{4}{5}$, $P(\omega_2|x) = \frac{1}{5}$, this object belongs to ω_1.

(b) We want to derive the value of x where $P(\omega_1|x)$= $P(\omega_2|x)$, which will be the boundary of the classifier. Due to the equal prior, we can instead solve:

$$P(x|\omega_1) = P(x|\omega_2)$$

x=0.8 is the boundary.

2. The boundary still means we need to get the x where $P(\omega_1|x)$= $P(\omega_2|x)$. Also, according to the probabilistic distribution we know that the boundary lies in the region of [0, 1], where:

$$P(x|\omega_1) = 1 - x$$

$$P(x|\omega_2) = \frac{1}{4}x$$

Applying Bayes rule, we have:

$$P(\omega_1|x) = \frac{P(\omega_1)\,P(x|\omega_1)}{P(x)} = \frac{P(\omega_1)\,P(x|\omega_1)}{P(\omega_1)\,P(x|\omega_1) + P(\omega_2)\,P(x|\omega_2)} = \frac{\frac{1}{3}(1-x)}{\frac{1}{3} - \frac{1}{6}x} = \frac{2-2x}{2-x}$$

Similarly,

$$P(\omega_2|x) = \frac{x}{2-x}$$

Applying

$$P(\omega_1|x) = \ P(\omega_2|x)$$

We have

$$\frac{2-2x}{2-x} = \frac{x}{2-x}$$

$$x = \frac{2}{3}$$

Therefore, $x = \frac{2}{3}$ is the boundary of the classifier.