Constructive solid geometry and Nef polyhedra

GEO1004:
3D modelling of the built environment

https://3d.bk.tudelft.nl/courses/geo1004

3D geoinformation
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Midterm answers

3D representations so far...

- Explicit geometries:
  - B-rep with meshes
  - 3D Delaunay tetrahedralisations / Voronoi diagrams
  - Voxels
  - MAT / skeleton
- Moving towards implicit geometries: curves

Explicit and implicit geometries

- Explicit: more direct descriptions, e.g. point coordinates or the equations typically used to describe shapes
- Implicit: more indirect descriptions, e.g. sequences of operations or complex functions

Why implicit geometries?

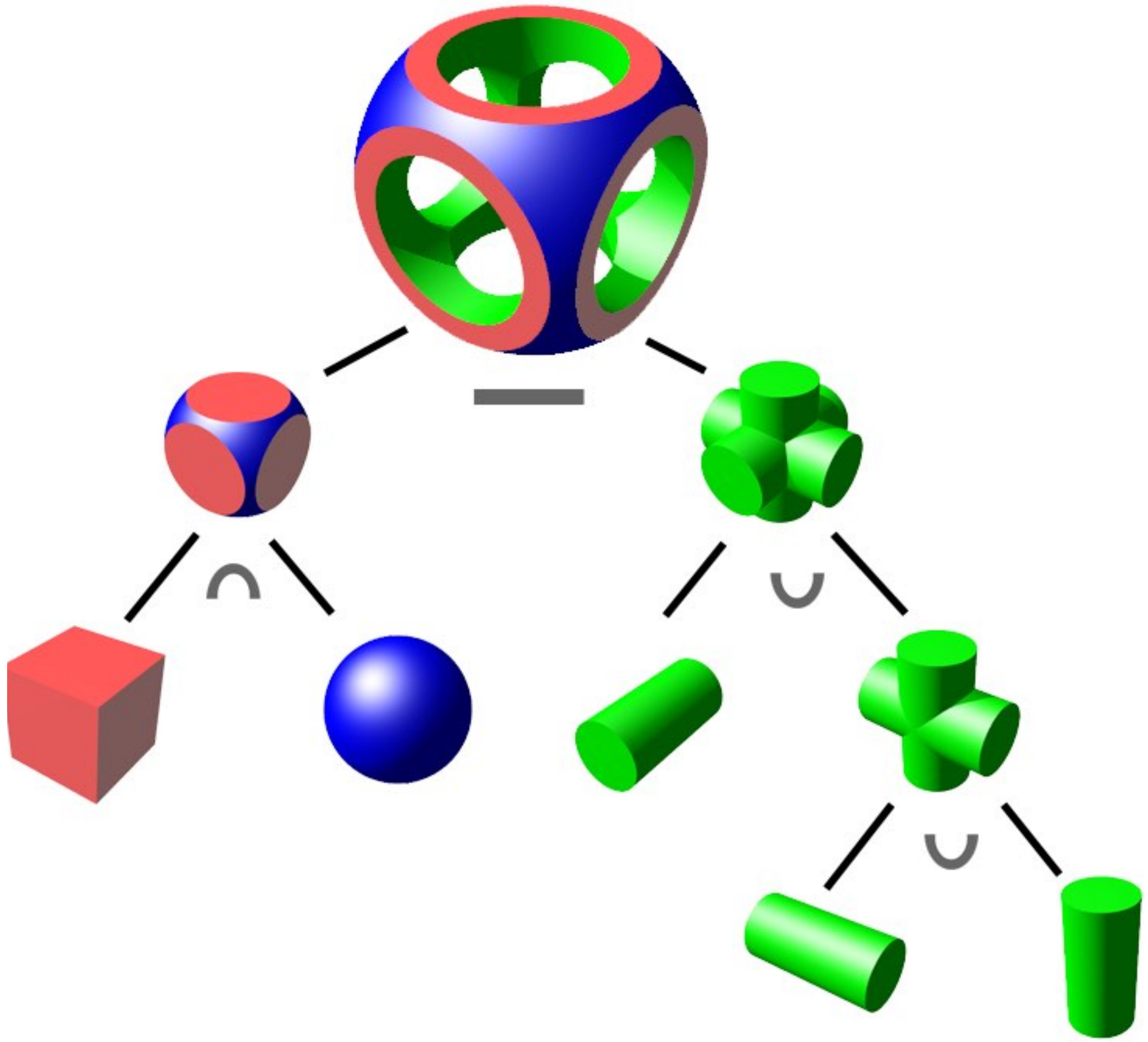
- Compact: complex shapes can be represented by a few functions rather than many primitives in a mesh
- Resolution independence: smooth shapes are always smooth, can be evaluated anywhere
- Convertibility: much easier to convert to explicit geometries than vice versa

Where implicit geometries?

- 3D modelling and animation, e.g. creation of models through 3D primitives
- BIM: essentially all building elements except terrain (next Wednesday)
- CAD and manufacturing
- ML / DL for 3D: many methods are based on implicit fields, e.g. occupancy

Constructive solid geometry

- In short: representing complex shapes as operations on simple shapes
  - operations: usually Boolean set ops + possibly some others
  - simple shapes: parametric shapes, half-spaces + maybe (some) polyhedra



Background

Sets

- Sets: collections of abstract objects called elements
  - {1, 2, 3} = {3, 2, 1}
- Elements can be anything: letters, numbers, symbols or other sets
- Defined using { and } in two ways:
  - Listing elements: {1, 2, 3}
  - Specifying rules: $\{x : x$ is a prime number$\}$

Set terms and notation

- Element a is in set $\mathbb{X}$: $a \in \mathbb{X}$
- Element a is not in set $\mathbb{X}$: $a \notin \mathbb{X}$
- And ($\wedge$), or ($\vee$) and not ($\neg$)
- Set with no elements: empty set, {} or $\emptyset$
- Set with all elements (within context): universe set or $\mathbb{U}$

Set operations

- Similar to $=$, $<$, $\leq$, $>$ and $\geq$:
  - $\mathbb{A} = \mathbb{B}$: sets $\mathbb{A}$ and $\mathbb{B}$ have the same elements
  - $\mathbb{A} \subseteq \mathbb{B}$: every element in $\mathbb{A}$ is in $\mathbb{B}$
  - $\mathbb{A} \subset \mathbb{B}$: every element in $\mathbb{A}$ is in $\mathbb{B}$ and also $\mathbb{B}$ has at least one more element that is not in $\mathbb{A}$

Tuples

- Tuples: ordered sequences of elements (unlike sets)
    - $(1, 2, 3) \neq (3, 2, 1)$
- Defined using (and)
- 2 elements: pair, 3 elements: treble/triplet, n elements: n-tuple

Cartesian product

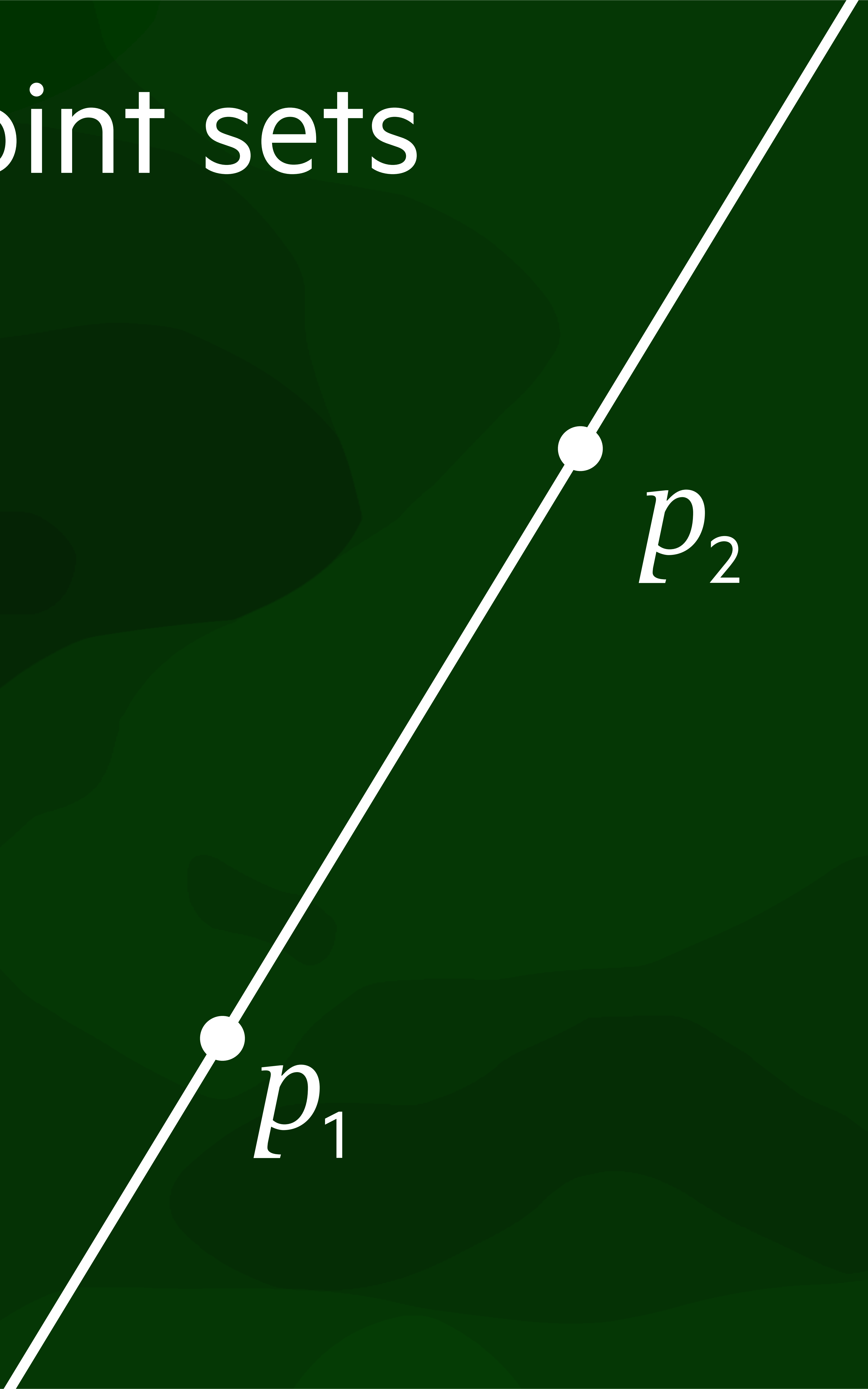
- Operation to build a set of tuples from sets
- $\mathbb{A} \times \mathbb{B} = \{(a, b) : a \in \mathbb{A} \wedge b \in \mathbb{B}\}$
  - Set of all possible tuples
  - ...where the first element is in $\mathbb{A}$
  - ...and the second element is in $\mathbb{B}$
- $\mathbb{A}^2 = \mathbb{A} \times \mathbb{A}$, $\mathbb{A}^3 = \mathbb{A} \times \mathbb{A} \times \mathbb{A}$, etc.

Point sets

- Set $\mathbb{R}$: all real numbers
- Using Cartesian geometry, it's possible to define space:
  - $\mathbb{R} = \{x : x \in \mathbb{R}\}$: 1D space (i.e. the line)
  - $\mathbb{R}^2 = \{(x, y) : x \in \mathbb{R} \wedge y \in \mathbb{R}\}$: 2D space (i.e. the plane)
  - $\mathbb{R}^3 = \{(x, y, z) : x \in \mathbb{R} \wedge y \in \mathbb{R} \wedge z \in \mathbb{R}\}$: 3D space

Objects as point sets

- Starting from two points p_1 and p_2
- The line L passing through the points is given by $\{t\,p_1 + (1-t)\,p_2 : t \in \mathbb{R}\}$
  - Similar to weighted average / linear interpolation of points
  - Works in any dimension



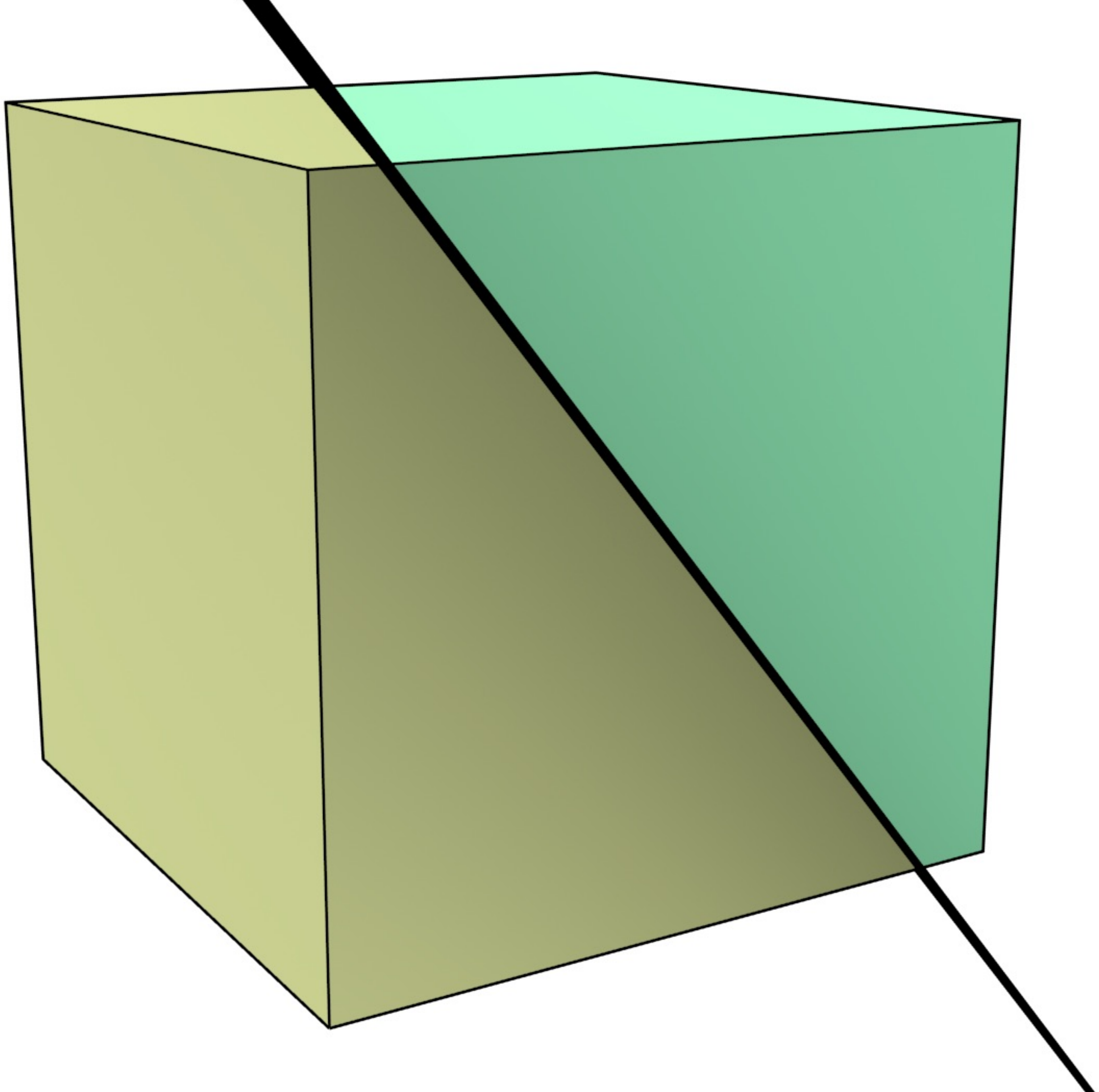
Objects as point sets

Similar parametric equations for many objects, such as a plane:

$$P = \left\{ \frac{ap_1 + bp_2 + cp_3}{a + b + c} : a, b, c \in \mathbb{R} \right\}$$

or half-space:

$$H \leq \left\{ \frac{ap_1 + bp_2 + cp_3}{a + b + c} : a, b, c \in \mathbb{R} \right\}$$



Objects as point sets

or a ball of radius r:

$$B = \left\{(x, y, z) : (x - c_x)^2 + (y - c_y)^2 + (z - c_z)^2 \leq r^2\right\}$$

or a cuboid (box):

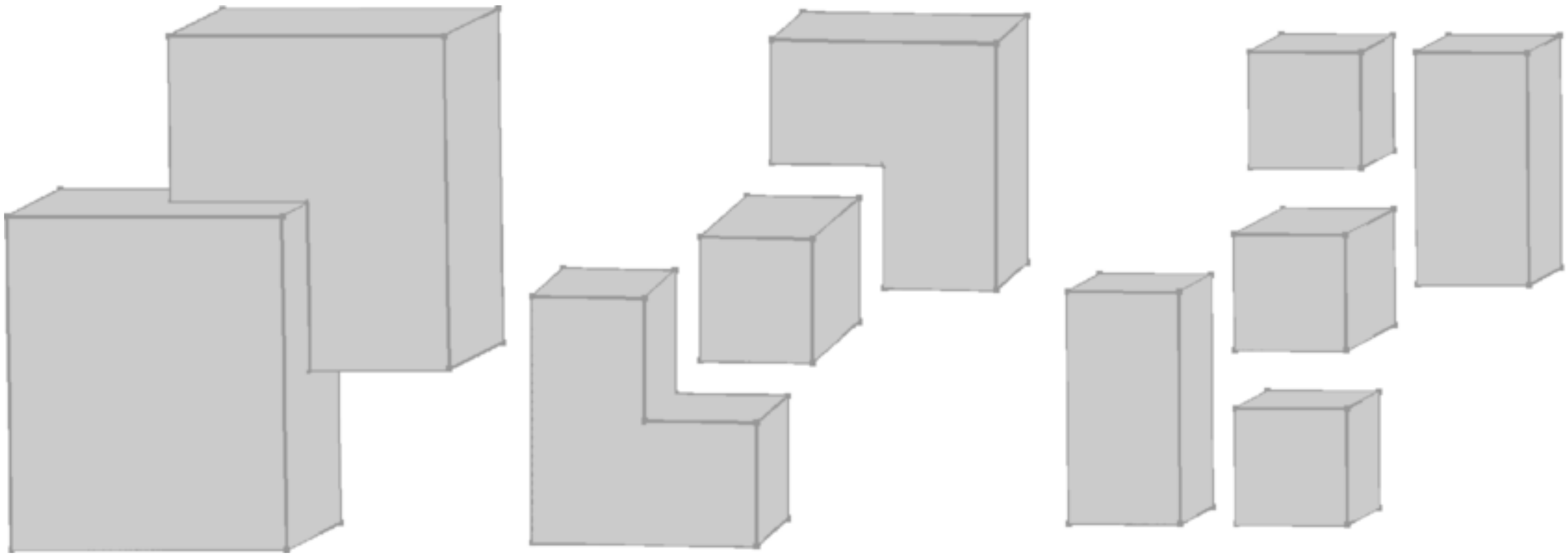
$$C = \left\{(x, y, z) : x_{\min} \leq x \leq x_{\max} \wedge y_{\min} \leq y \leq y_{\max} \wedge z_{\min} \leq z \leq z_{\max}\right\}$$

Polyhedra

- Arbitrary polyhedra can be represented as...

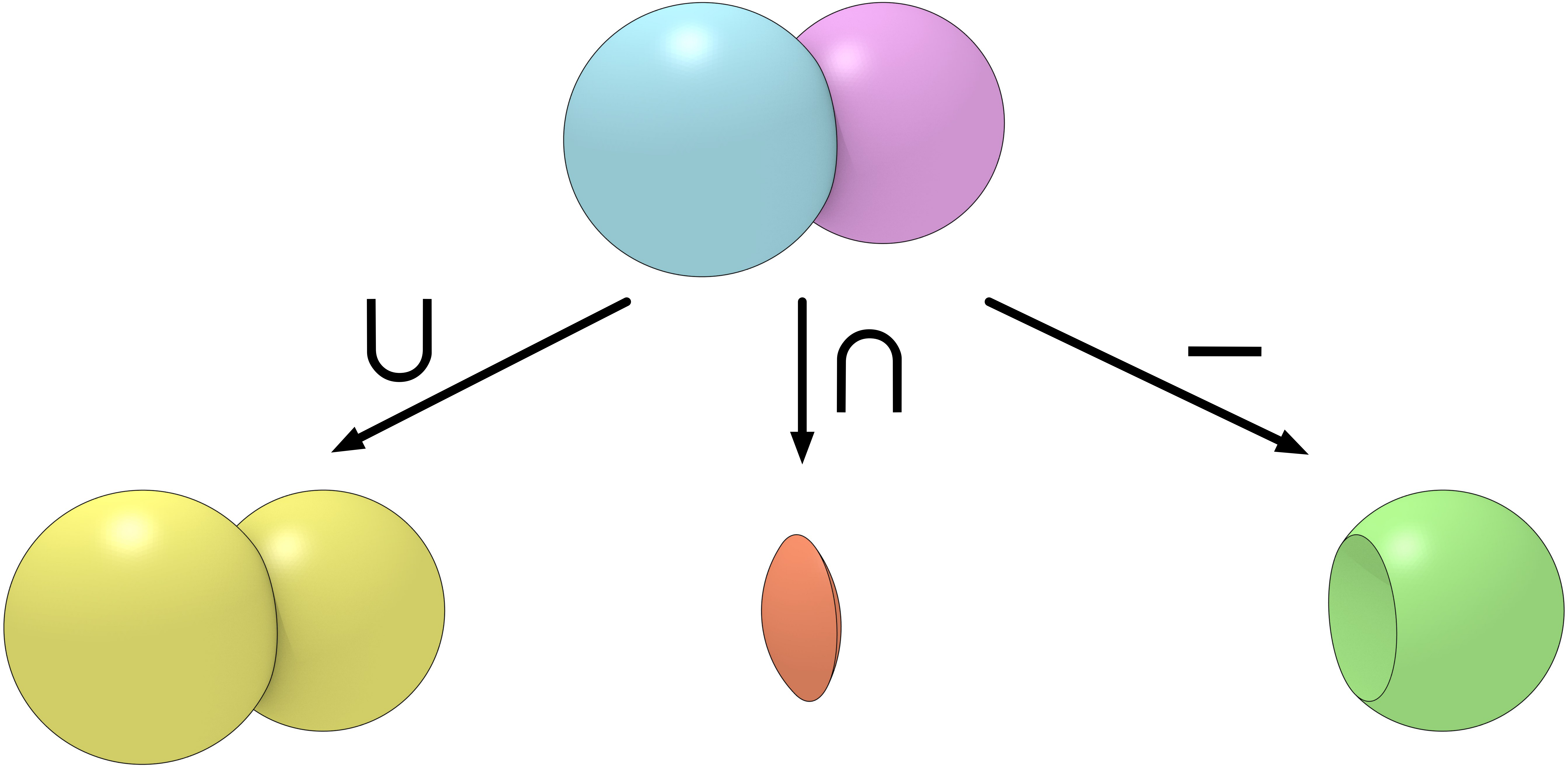
- Union of convex polyhedra (through convex decomposition), each of which is represented as:

- Intersection of half-spaces



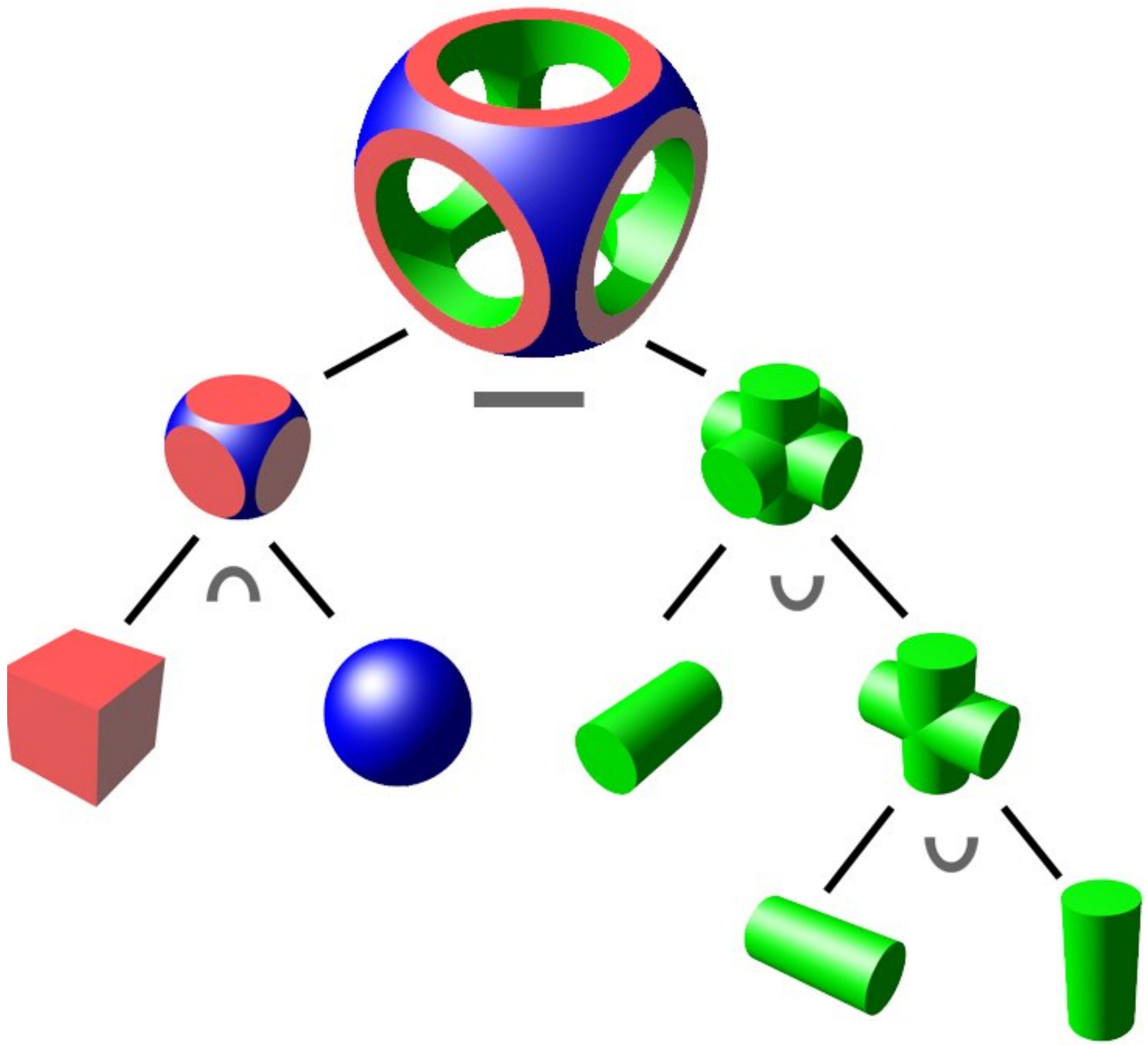
Boolean set operations

- Union: $\mathbb{A} \cup \mathbb{B} = \{x : x \in \mathbb{A} \vee x \in \mathbb{B}\}$ (elements that are in $\mathbb{A}$ or in $\mathbb{B}$)
- Intersection: $\mathbb{A} \cap \mathbb{B} = \{x : x \in \mathbb{A} \wedge x \in \mathbb{B}\}$ (elements that are in $\mathbb{A}$ and in $\mathbb{B}$)
- Difference: $\mathbb{A} - \mathbb{B} = \{x : x \in \mathbb{A} \wedge x \notin \mathbb{B}\}$ (elements that are in $\mathbb{A}$ but not in $\mathbb{B}$)
- Complement: $\neg\mathbb{A} = \{x : x \in \mathbb{U} \wedge x \notin \mathbb{A}\}$ (elements that are not in $\mathbb{A}$)



A CSG engine

- Leaf nodes: primitives defined as point sets
- Non-leaf nodes: Boolean set operations that operate on point sets
- Combined geometries are just evaluations on point sets, e.g. on a voxel grid of arbitrary resolution

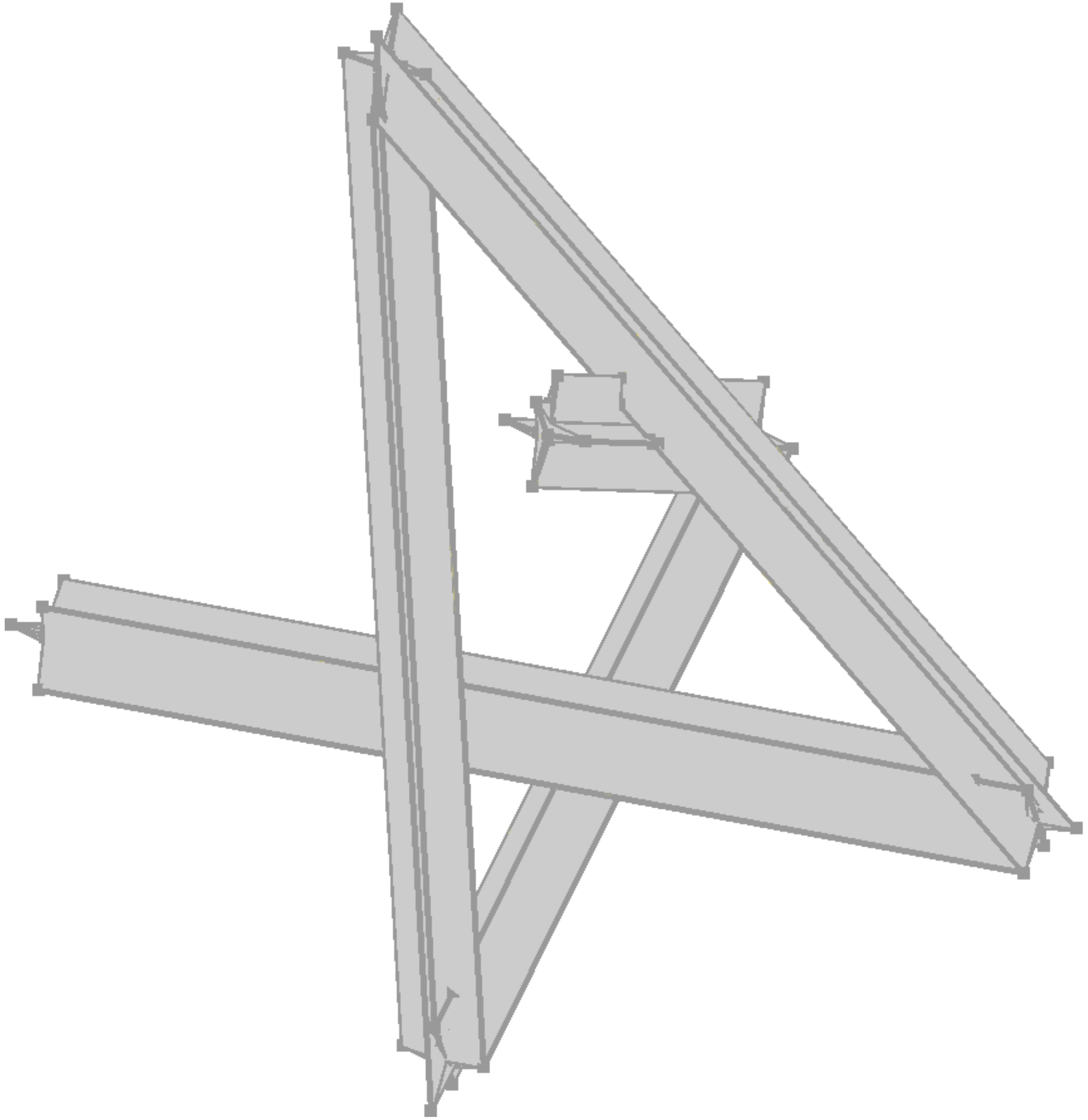


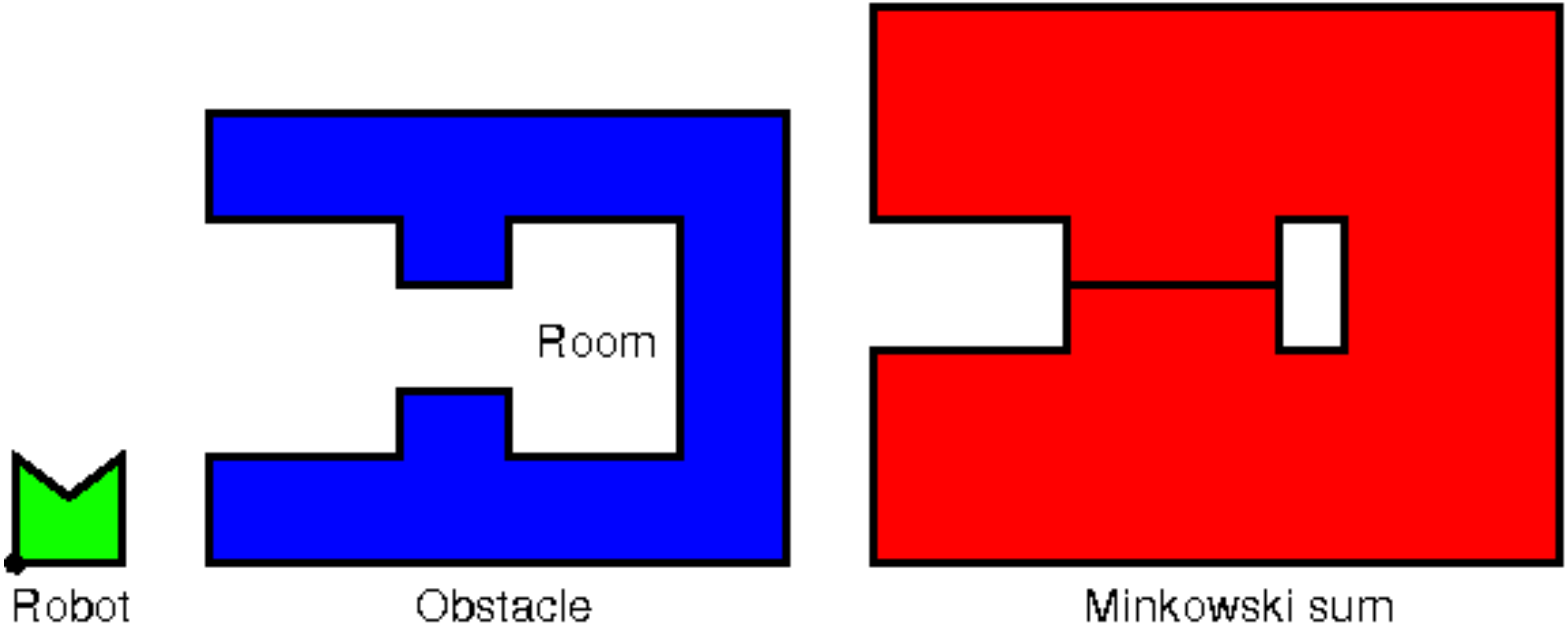
Minkowski sums

- Starting from two point sets P and Q
- P ⊕ Q = $\{p + q \mid p \in P, q \in Q\}$

Minkowski sums

- Sweeps: path ⊕ 2D profile or 3D shape



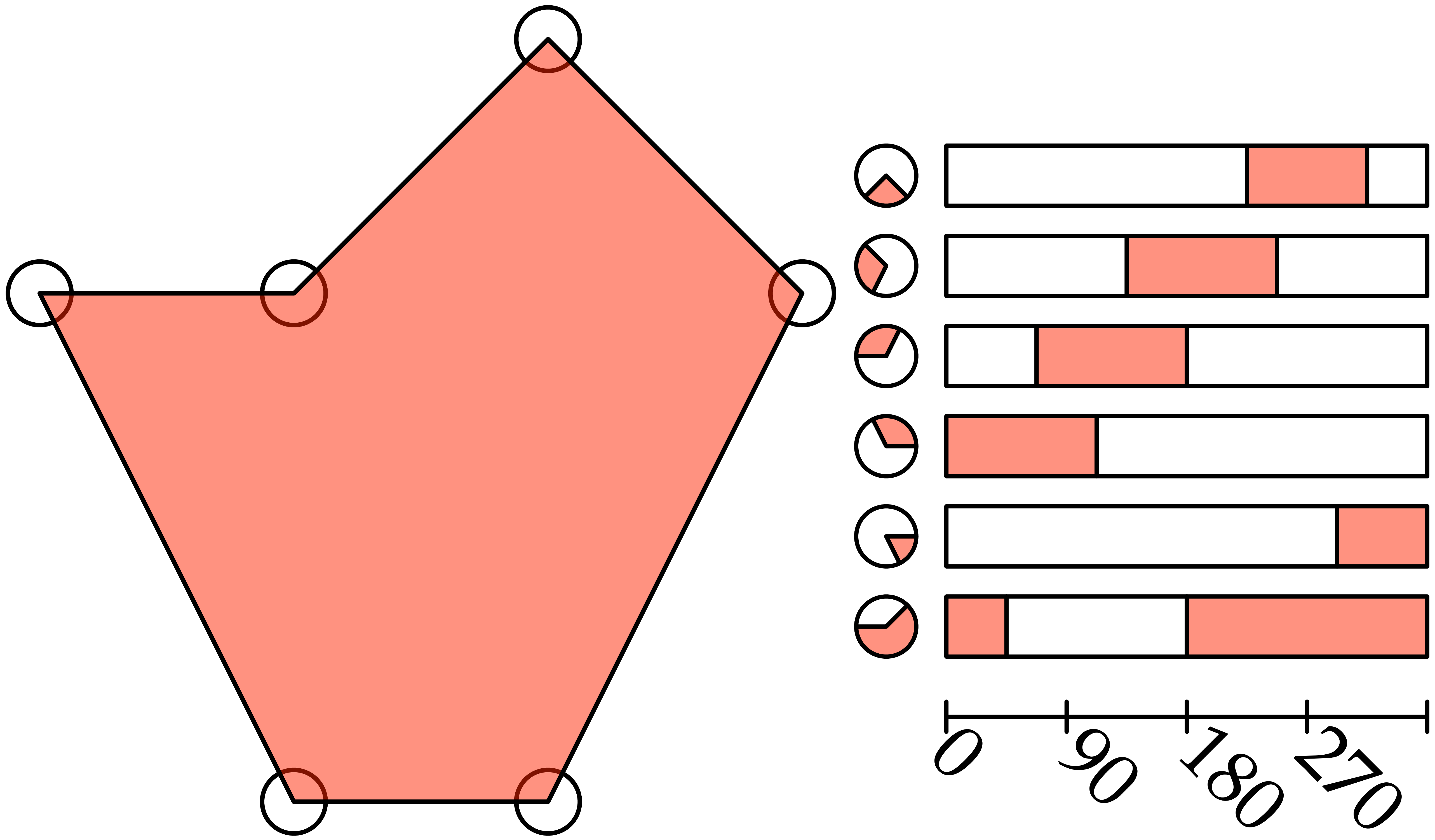

Room
Robot
Obstacle
Minkowski sum

Nef polyhedra

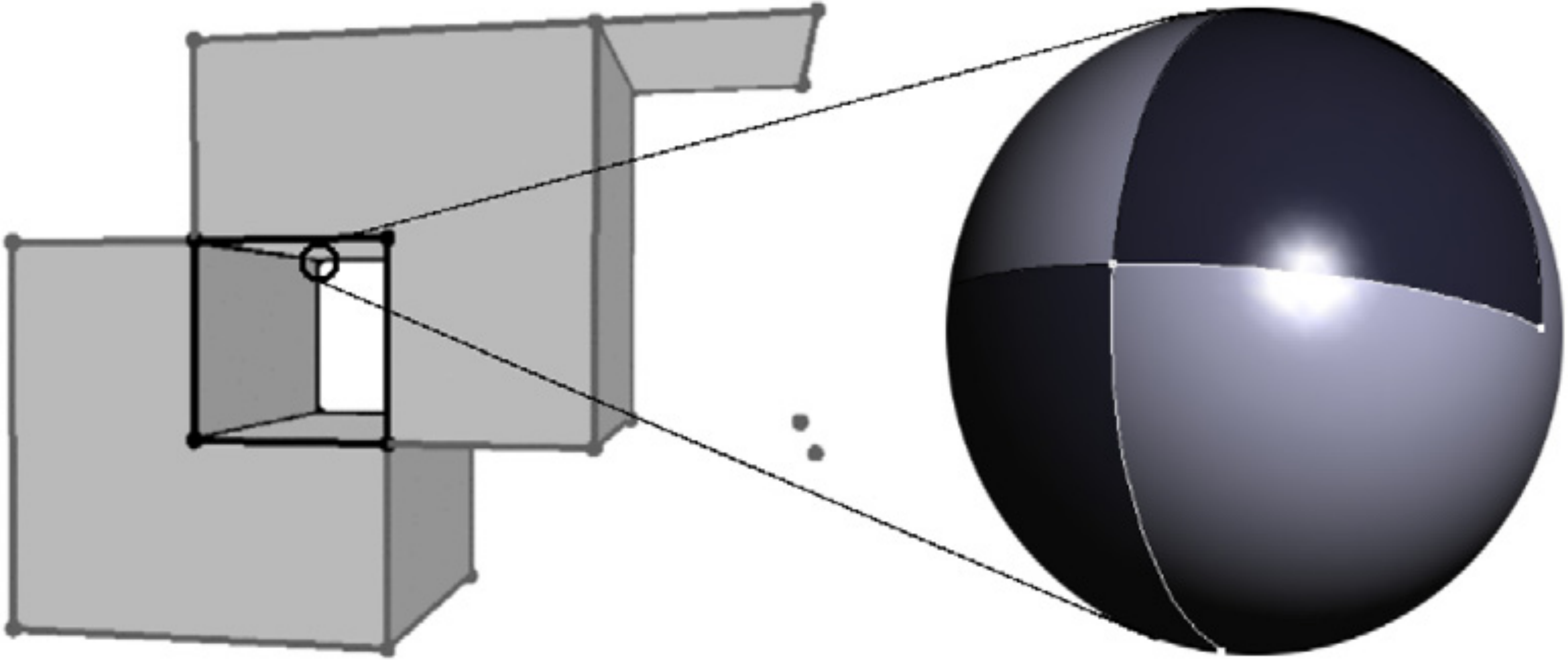
- Alternative definition of polyhedra with:
  - non-manifolds
  - robust Boolean point set operations
- Based on local pyramids
- Operations can be performed at the local pyramid level

Local pyramids

- Intersection of an infinitesimally small circle (2D) or sphere (3D) located at each vertex
- Dimension reduction mechanism in Nef polyhedra (akin to b-rep):
  - 2D Polygon as a set of vertices + 1D ranges
  - 3D Polyhedron as a set of vertices + 2D sphere map

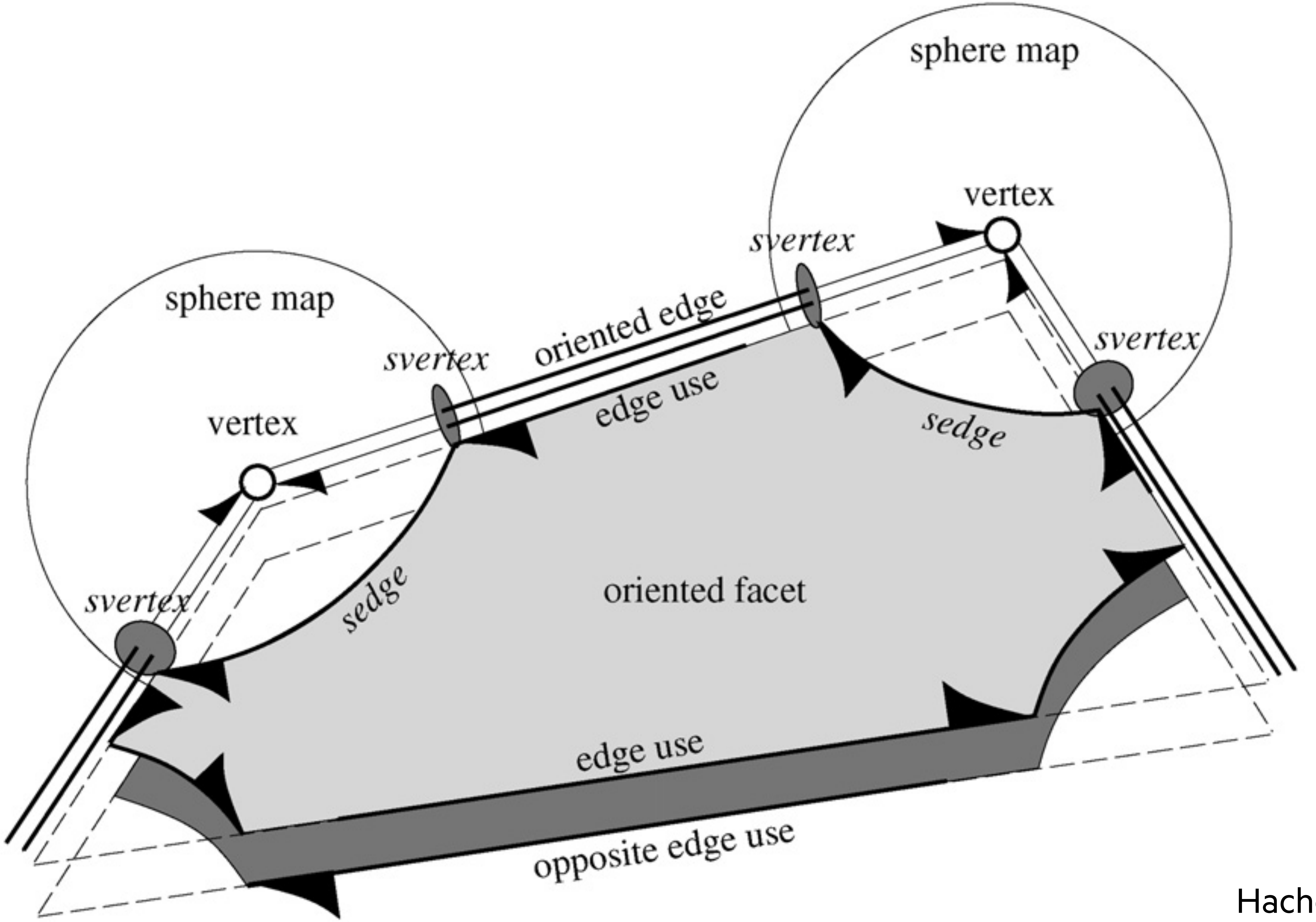


Local pyramids



Hachenberger et al. (2006)

Local pyramids

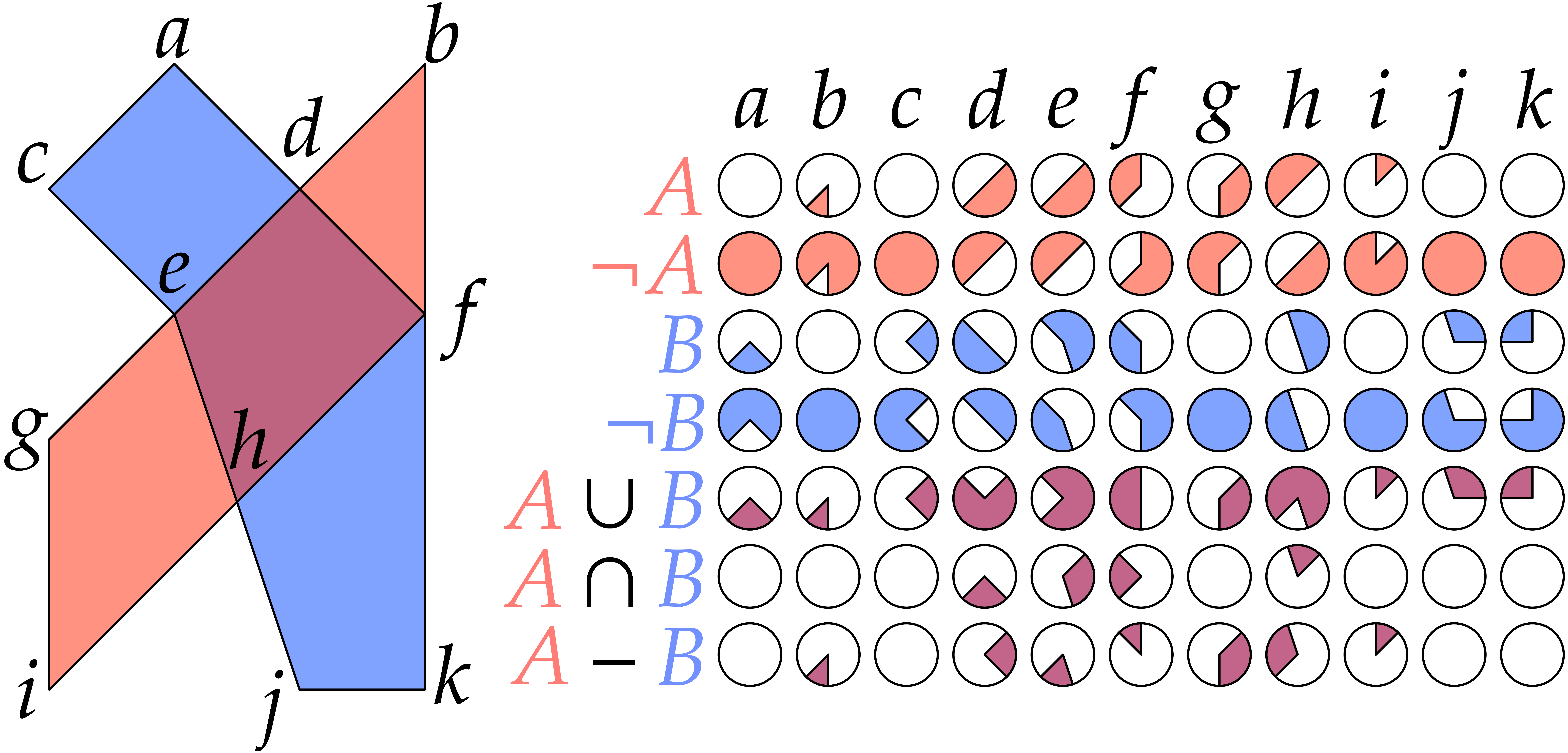


Hachenberger et al. (2006)

Operations on local pyramids

1. subdivision: compute overlay -> new local pyramids
2. selection: perform operation on local pyramids
3. simplification: remove unnecessary local pyramids

Operations on local pyramids



What to do next?

1. Today:
    - Continue with Homework 2 (due June 1)
    - Go to geo1004 website and study today's lesson (3D book Chapter 5)
2. Monday: public holiday
3. Wednesday: lecture about BIM with guest mini-lecture, then intro to Homework 3

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References

- Peter Hachenberger, Lutz Kettner and Kurt Mehlhorn. **Boolean operations on 3D selective Nef complexes: Data structure, algorithms, optimized implementation and experiments**. Computational Geometry 38 (1-2) pp. 64-99. 2007.
- CGAL packages: 3D Boolean Operations on Nef Polyhedra, Convex Decomposition of Polyhedra and 3D Minkowski Sum of Polyhedra.