

# Constructive solid geometry and Nef polyhedra

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GEO1004:  
3D modelling of the built environment

<https://3d.bk.tudelft.nl/courses/geo1004>



3D geoinformation

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# 3D representations so far...

- Explicit geometries:
  - 3D objects through 2D boundary (b-rep + meshes)
  - Voxels
  - 3D Delaunay tetrahedralisations / Voronoi diagrams
  - MAT / skeleton

# Explicit and implicit geometries

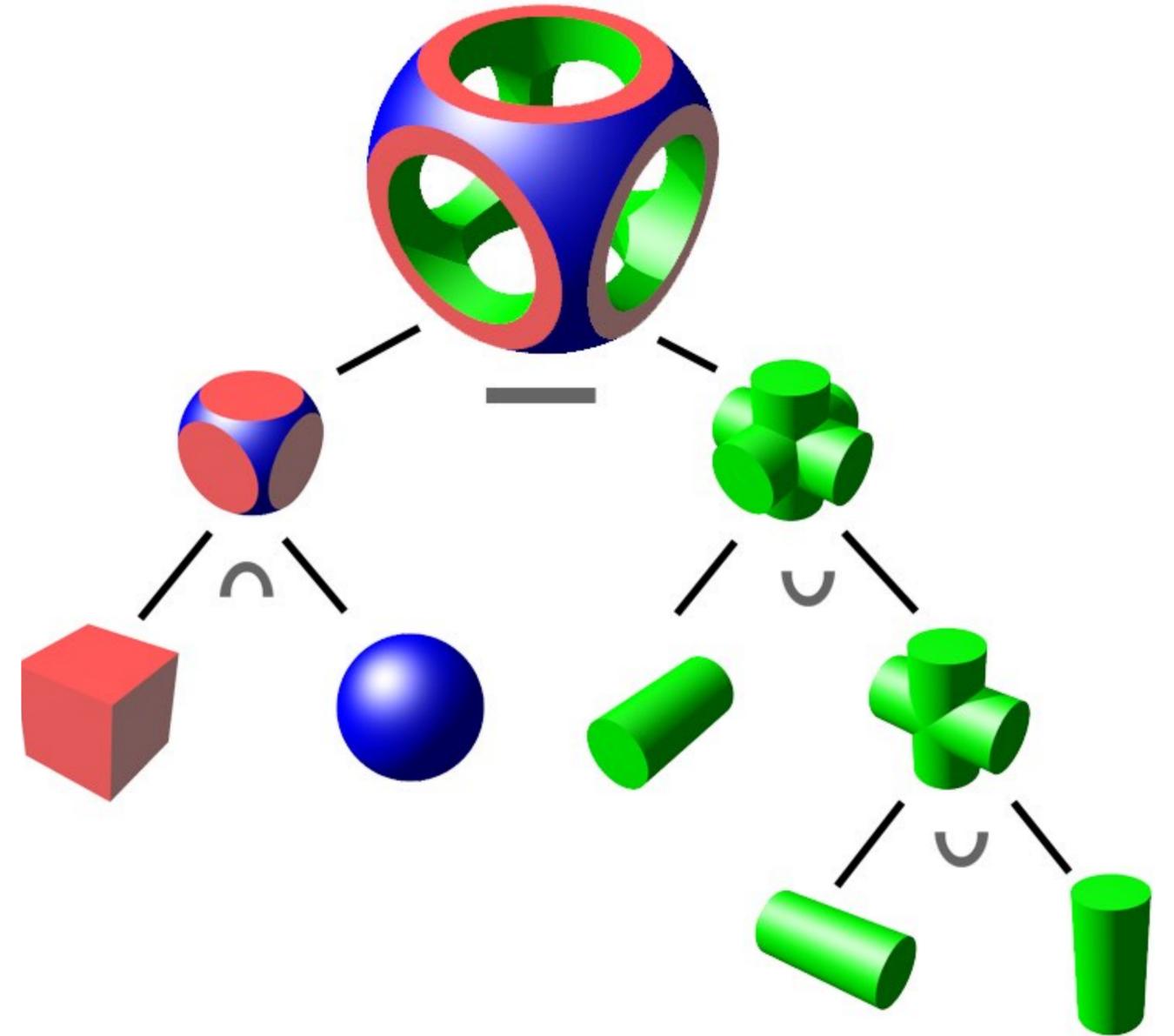
- Explicit: more direct descriptions, e.g. point coordinates or the equations typically used to describe shapes
- Implicit: more indirect descriptions, e.g. sequences of operations or complex functions

# Why implicit geometries?

- Compact: complex shapes can be represented by a few functions rather than many primitives in a mesh
- Resolution independence: smooth shapes are always smooth, can be evaluated anywhere
- Convertibility: much easier to convert to explicit geometries than vice versa

# Constructive solid geometry

- In short: representing complex shapes as operations on simple shapes
- operations: usually Boolean set ops + possibly some others
- simple shapes: parametric shapes, half-spaces + maybe (some) polyhedra



Background

# Sets

- Sets: collections of abstract objects called elements
  - $\{1, 2, 3\} = \{3, 2, 1\}$
- Elements can be anything: letters, numbers, symbols or other sets
- Defined using { and } in two ways:
  - Listing elements:  $\{1, 2, 3\}$
  - Specifying rules:  $\{x : x \text{ is a prime number}\}$

# Set terms and notation

- Element  $a$  is in set  $X$ :  $a \in X$
- Element  $a$  is not in set  $X$ :  $a \notin X$
- And ( $\wedge$ ), or ( $\vee$ ) and not ( $\neg$ )
- Set with no elements: empty set,  $\{\}$  or  $\emptyset$
- Set with all elements (within context): universe set or  $\mathbb{U}$

# Set operations

- Similar to  $=$ ,  $<$ ,  $\leq$ ,  $>$  and  $\geq$ :
  - $\mathbb{A} = \mathbb{B}$ : sets  $\mathbb{A}$  and  $\mathbb{B}$  have the same elements
  - $\mathbb{A} \subseteq \mathbb{B}$ : every element in  $\mathbb{A}$  is in  $\mathbb{B}$
  - $\mathbb{A} \subset \mathbb{B}$ : every element in  $\mathbb{A}$  is in  $\mathbb{B}$  and also  $\mathbb{B}$  has at least one more element that is not in  $\mathbb{A}$

# Tuples

- Tuples: ordered sequences of elements (unlike sets)
  - $(1, 2, 3) \neq (3, 2, 1)$
- Defined using ( and )
- 2 elements: pair, 3 elements: treble/triplet,  $n$  elements:  $n$ -tuple

# Cartesian product

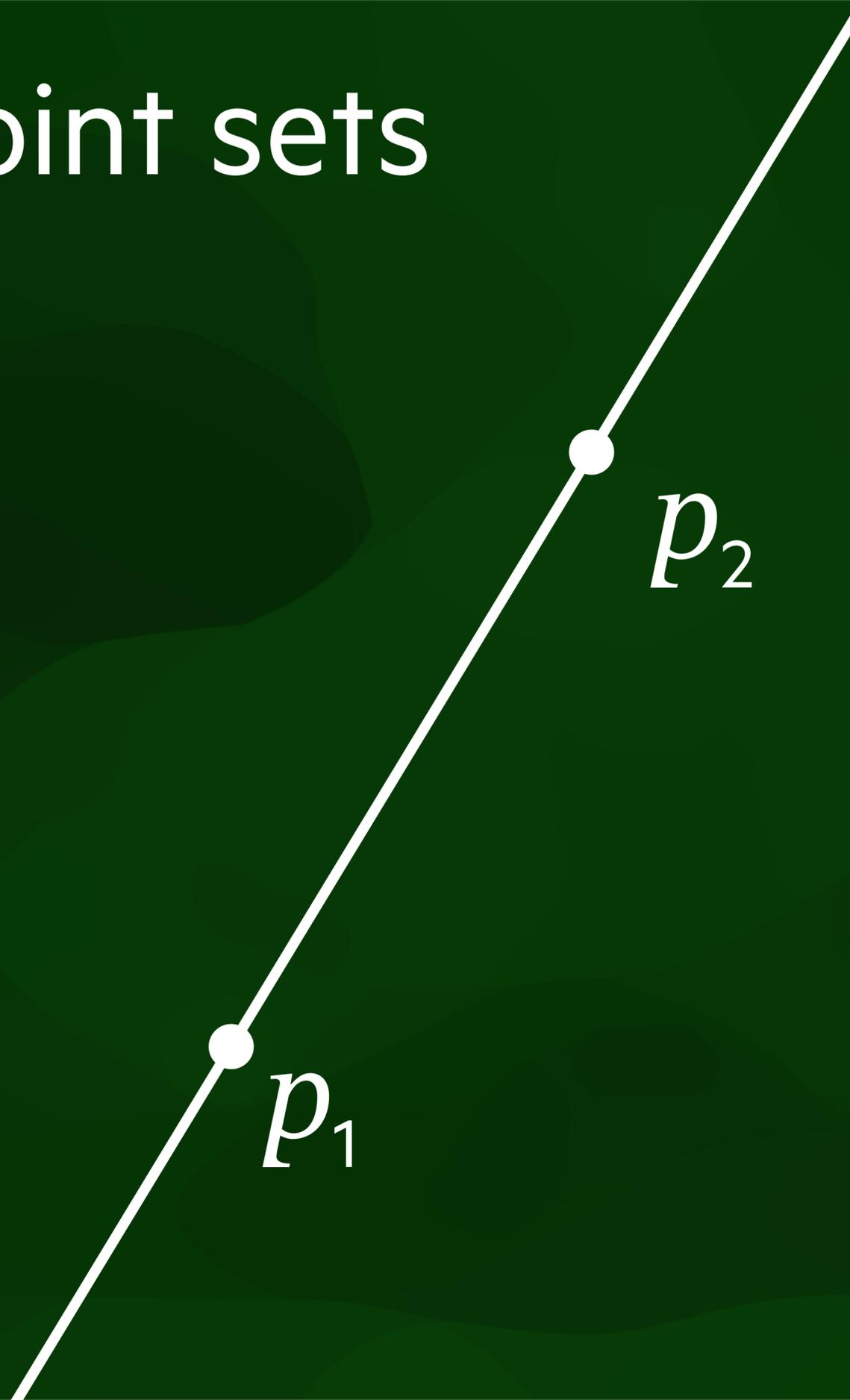
- Operation to build a set of tuples from sets
- $\mathbb{A} \times \mathbb{B} = \{(a, b) : a \in \mathbb{A} \wedge b \in \mathbb{B}\}$ 
  - Set of all possible tuples
  - ...where the first element is in  $\mathbb{A}$
  - ...and the second element is in  $\mathbb{B}$
- $\mathbb{A}^2 = \mathbb{A} \times \mathbb{A}$ ,  $\mathbb{A}^3 = \mathbb{A} \times \mathbb{A} \times \mathbb{A}$ , etc.

# Point sets

- Set  $\mathbb{R}$ : all real numbers
- Using Cartesian geometry, it's possible to define space:
  - $\mathbb{R} = \{x : x \in \mathbb{R}\}$ : 1D space (i.e. the line)
  - $\mathbb{R}^2 = \{(x, y) : x \in \mathbb{R} \wedge y \in \mathbb{R}\}$ : 2D space (i.e. the plane)
  - $\mathbb{R}^3 = \{(x, y, z) : x \in \mathbb{R} \wedge y \in \mathbb{R} \wedge z \in \mathbb{R}\}$ : 3D space

# Objects as point sets

- Starting from two points  $p_1$  and  $p_2$
- The line  $L$  passing through the points is given by  $\{tp_1 + (1-t)p_2 : t \in \mathbb{R}\}$
- Similar to weighted average / linear interpolation of points
- Works in any dimension



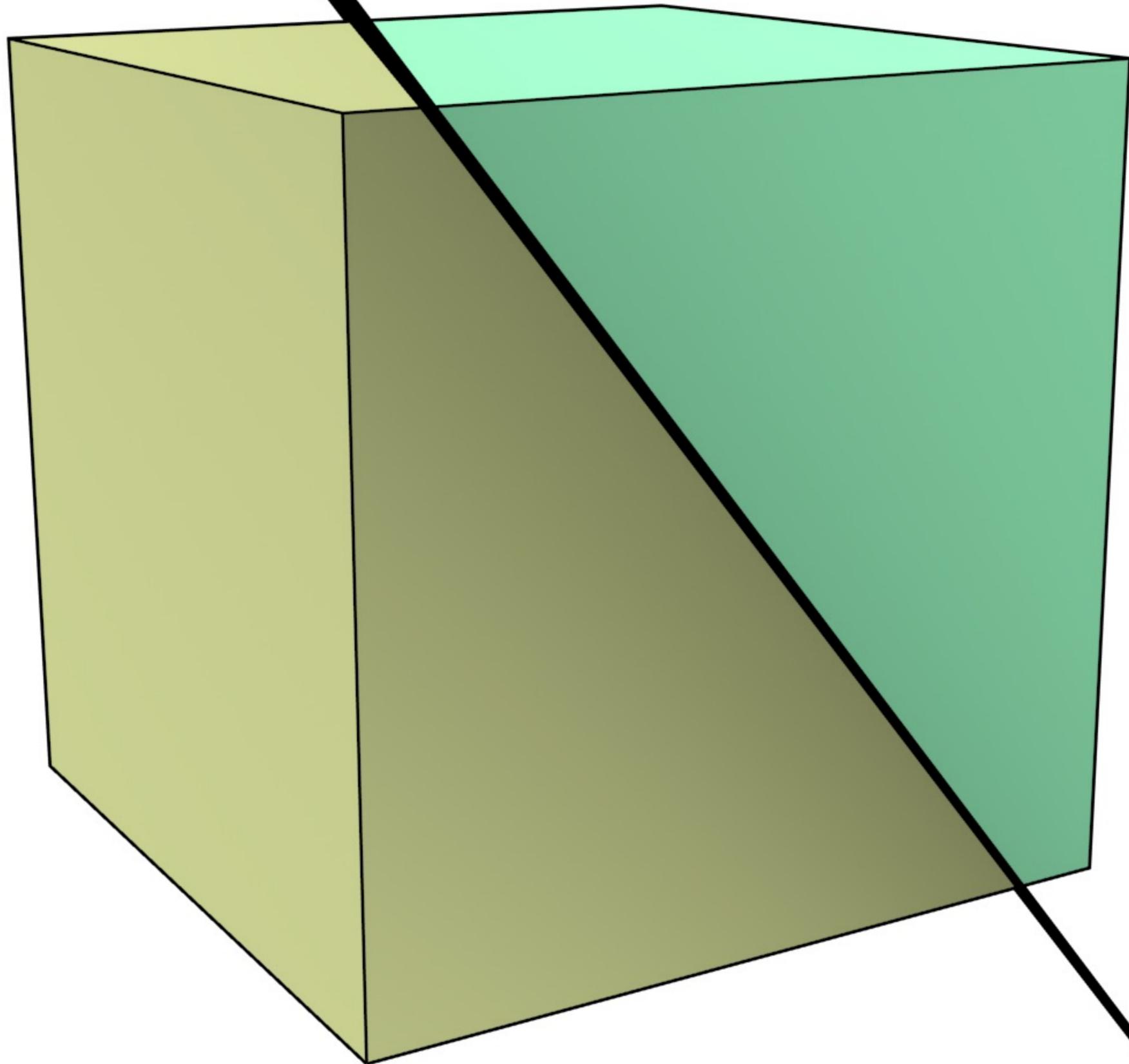
# Objects as point sets

Similar parametric equations for many objects, such as a plane:

$$P = \left\{ \frac{ap_1 + bp_2 + cp_3}{a + b + c} : a, b, c \in \mathbb{R} \right\}$$

or half-space:

$$H \leq \left\{ \frac{ap_1 + bp_2 + cp_3}{a + b + c} : a, b, c \in \mathbb{R} \right\}$$



# Objects as point sets

or a ball of radius  $r$ :

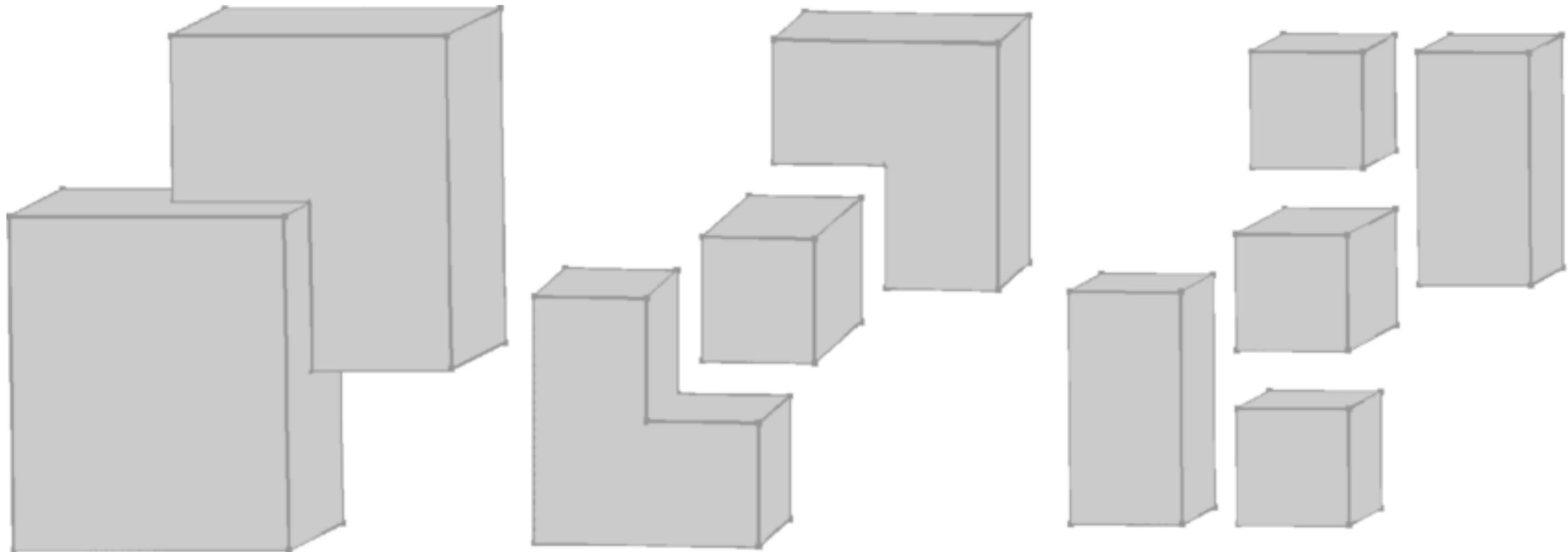
$$B = \{(x, y, z) : (x - c_x)^2 + (y - c_y)^2 + (z - c_z)^2 \leq r^2\}$$

or a cuboid (box):

$$C = \{(x, y, z) : x_{\min} \leq x \leq x_{\max} \wedge y_{\min} \leq y \leq y_{\max} \wedge z_{\min} \leq z \leq z_{\max}\}$$

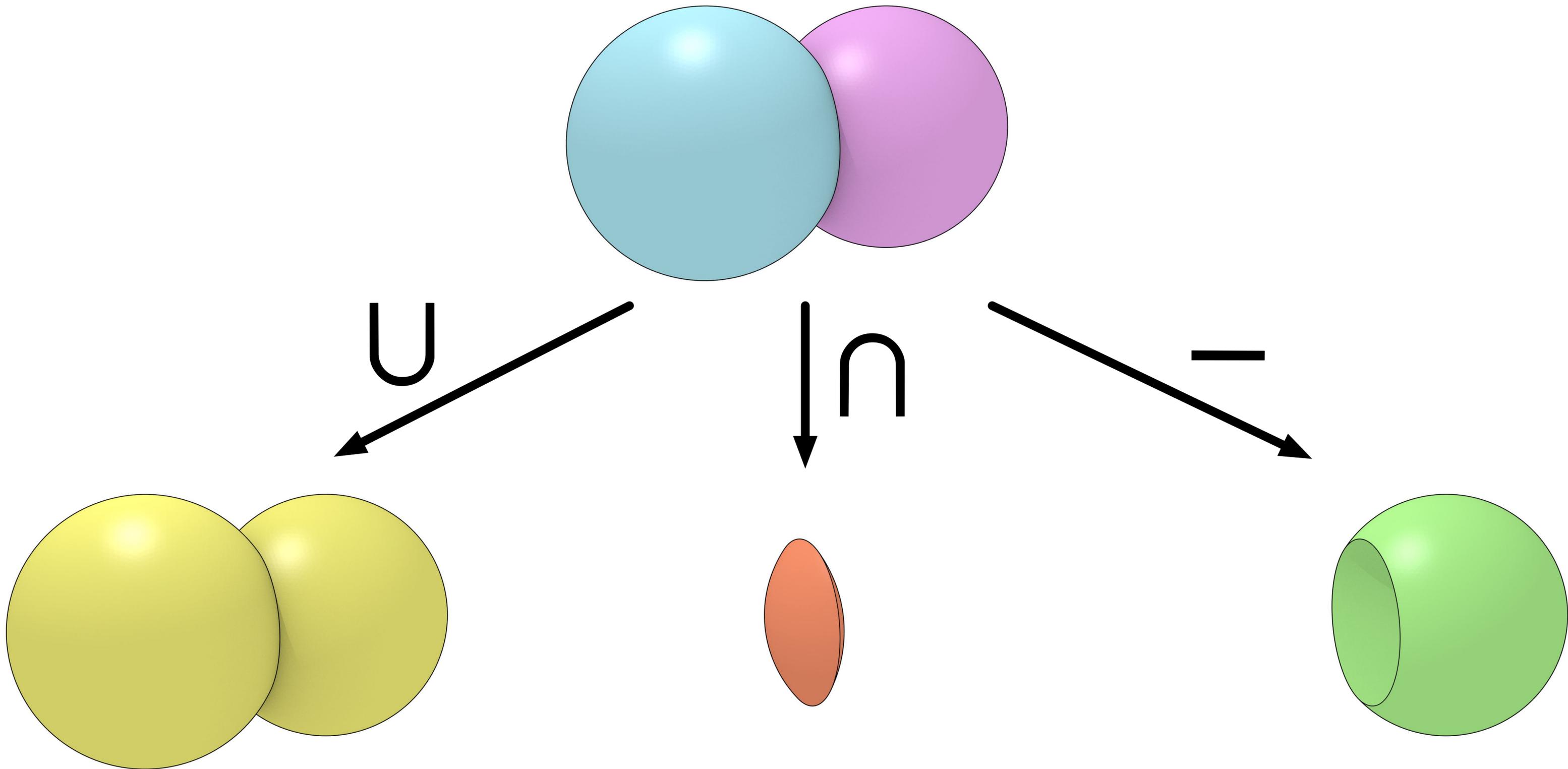
# Polyhedra

- Arbitrary polyhedra can be represented as...
- Union of convex polyhedra (through convex decomposition), each of which is represented as:
- Intersection of half-spaces



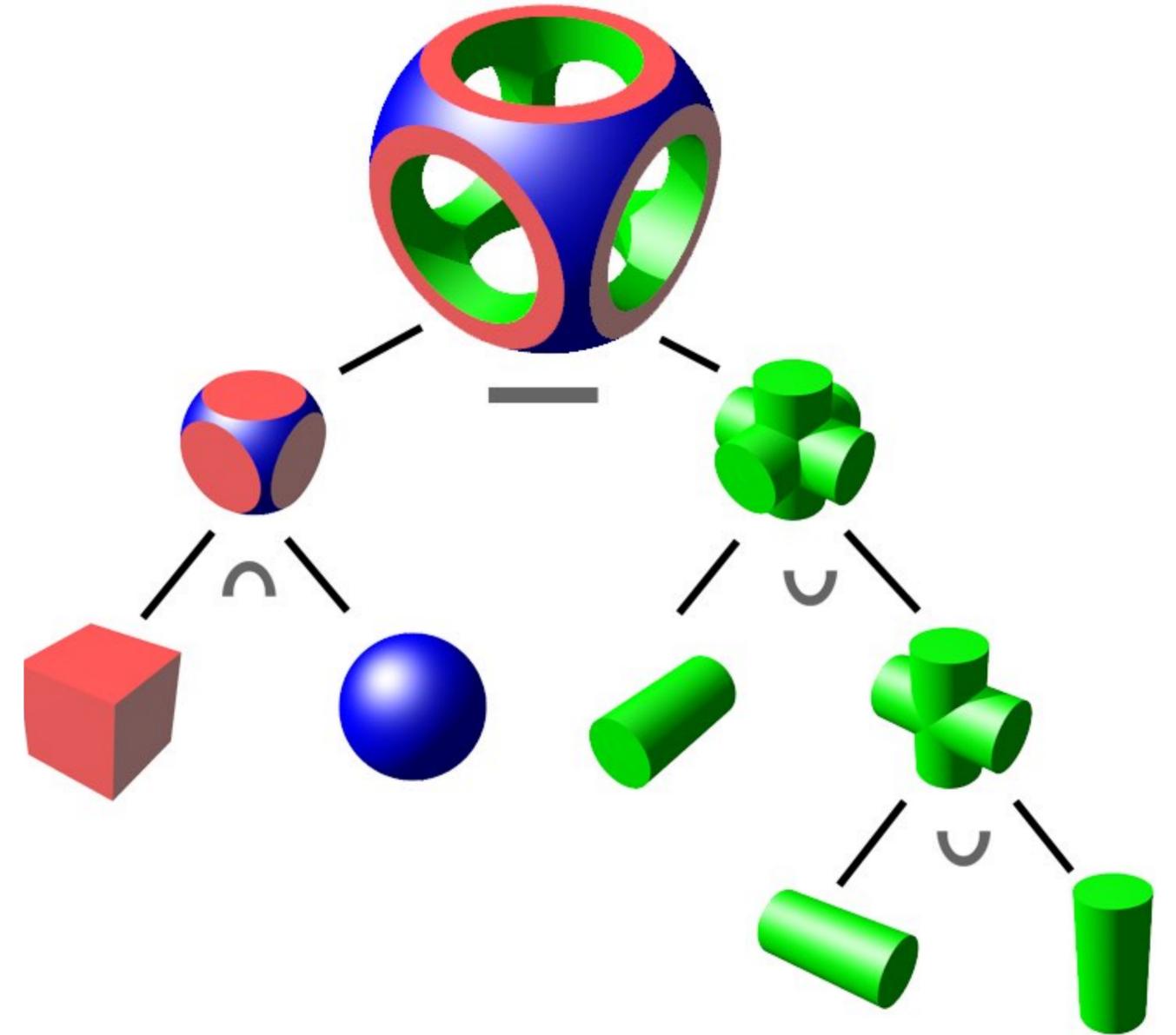
# Boolean set operations

- Union:  $A \cup B = \{x : x \in A \vee x \in B\}$  (elements that are in  $A$  or in  $B$ )
- Intersection:  $A \cap B = \{x : x \in A \wedge x \in B\}$  (elements that are in  $A$  and in  $B$ )
- Difference:  $A - B = \{x : x \in A \wedge x \notin B\}$  (elements that are in  $A$  but not in  $B$ )
- Complement:  $\neg A = \{x : x \in U \wedge x \notin A\}$  (elements that are not in  $A$ )



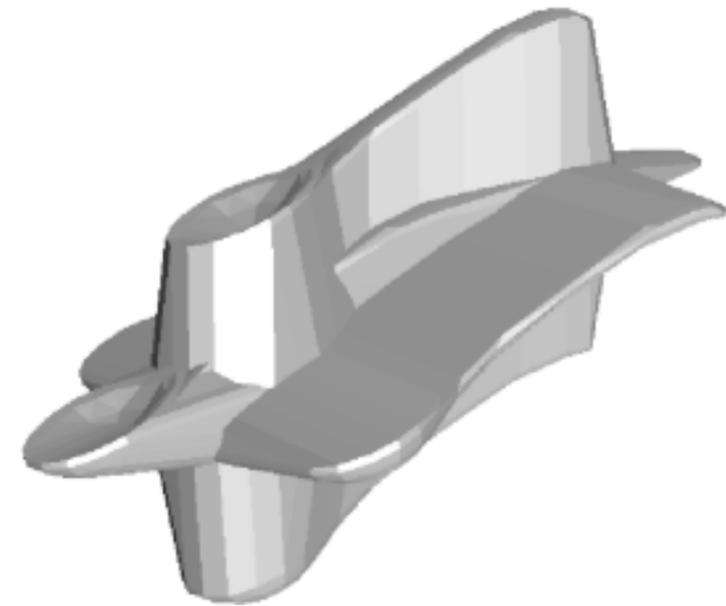
# A CSG engine

- Leaf nodes: primitives defined as point sets
- Non-leaf nodes: Boolean set operations that operate on point sets
- Combined geometries are just evaluations on point sets, e.g. on a voxel grid of arbitrary resolution



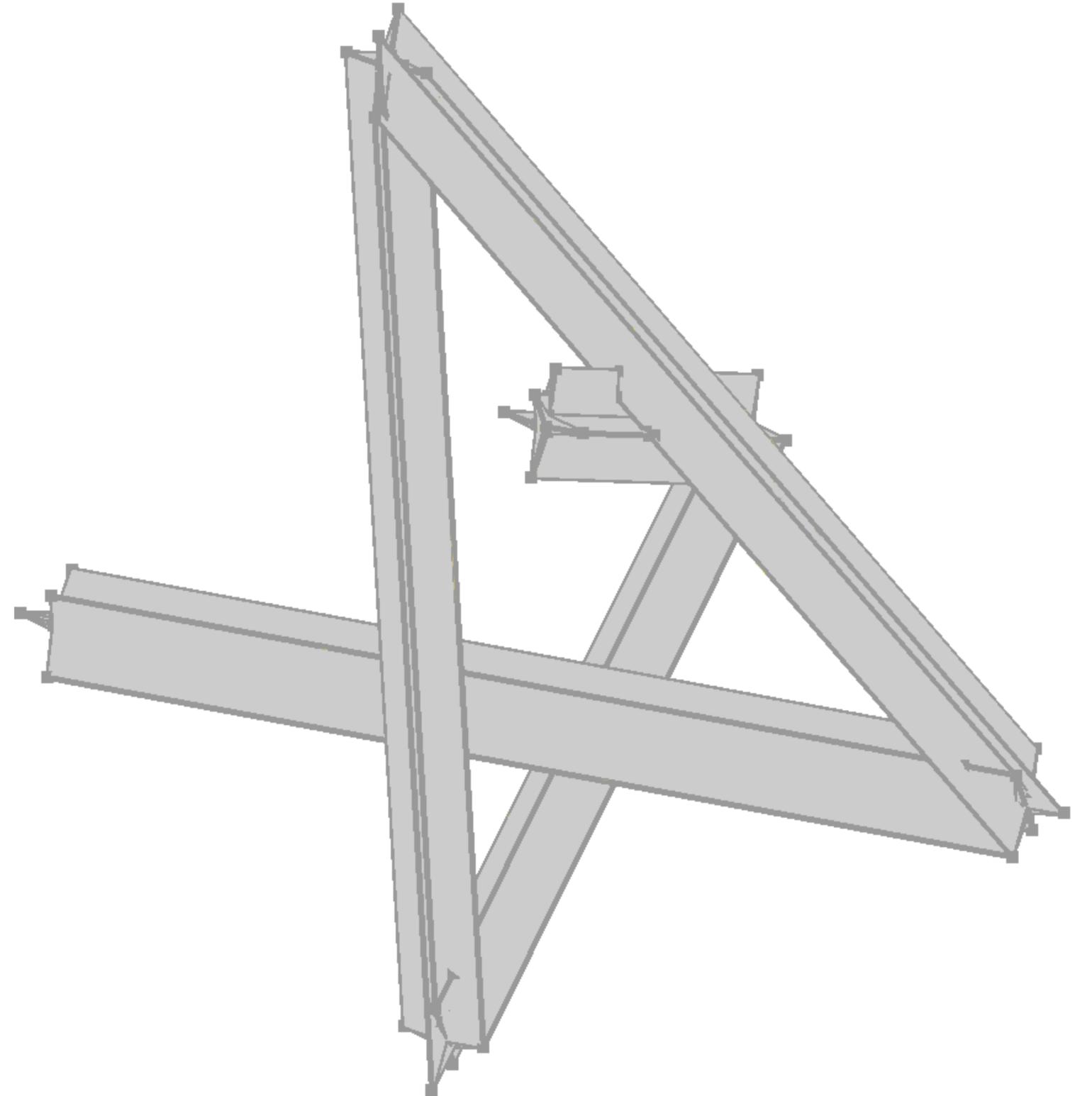
# Minkowski sums

- Starting from two point sets  $P$  and  $Q$
- $P \oplus Q = \{p + q \mid p \in P, q \in Q\}$



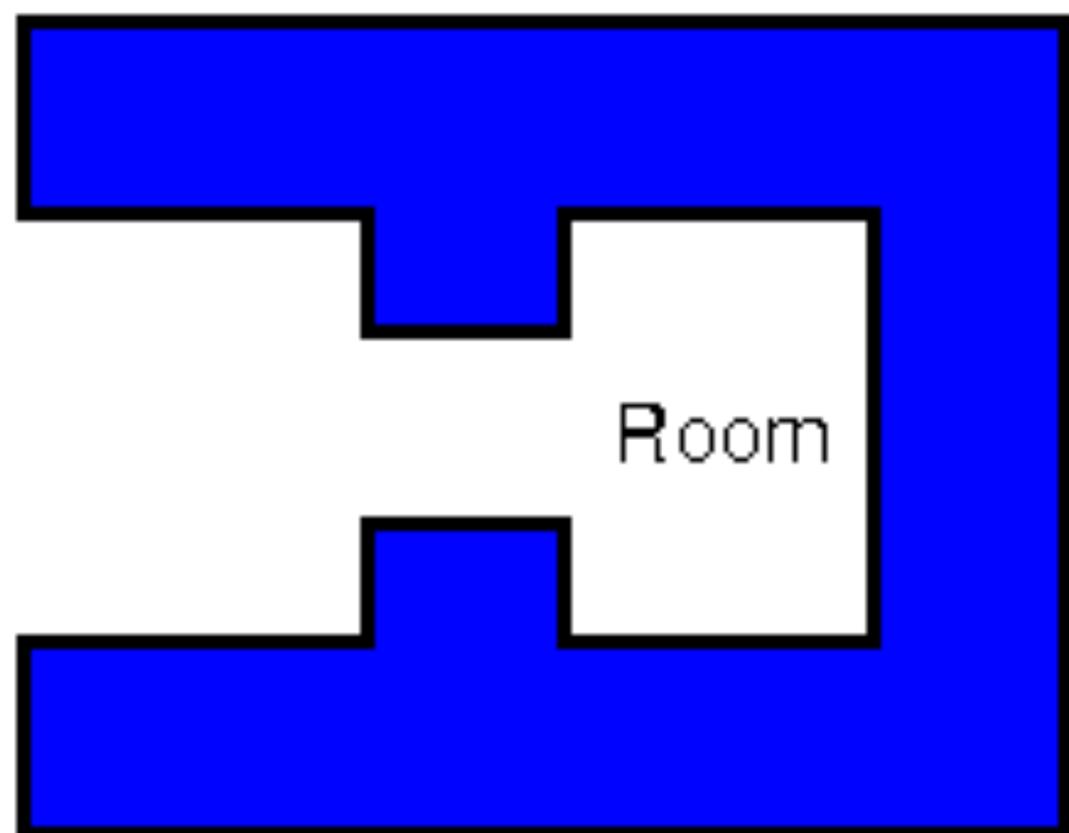
# Minkowski sums

- Sweeps: path  $\oplus$  2D profile or 3D shape



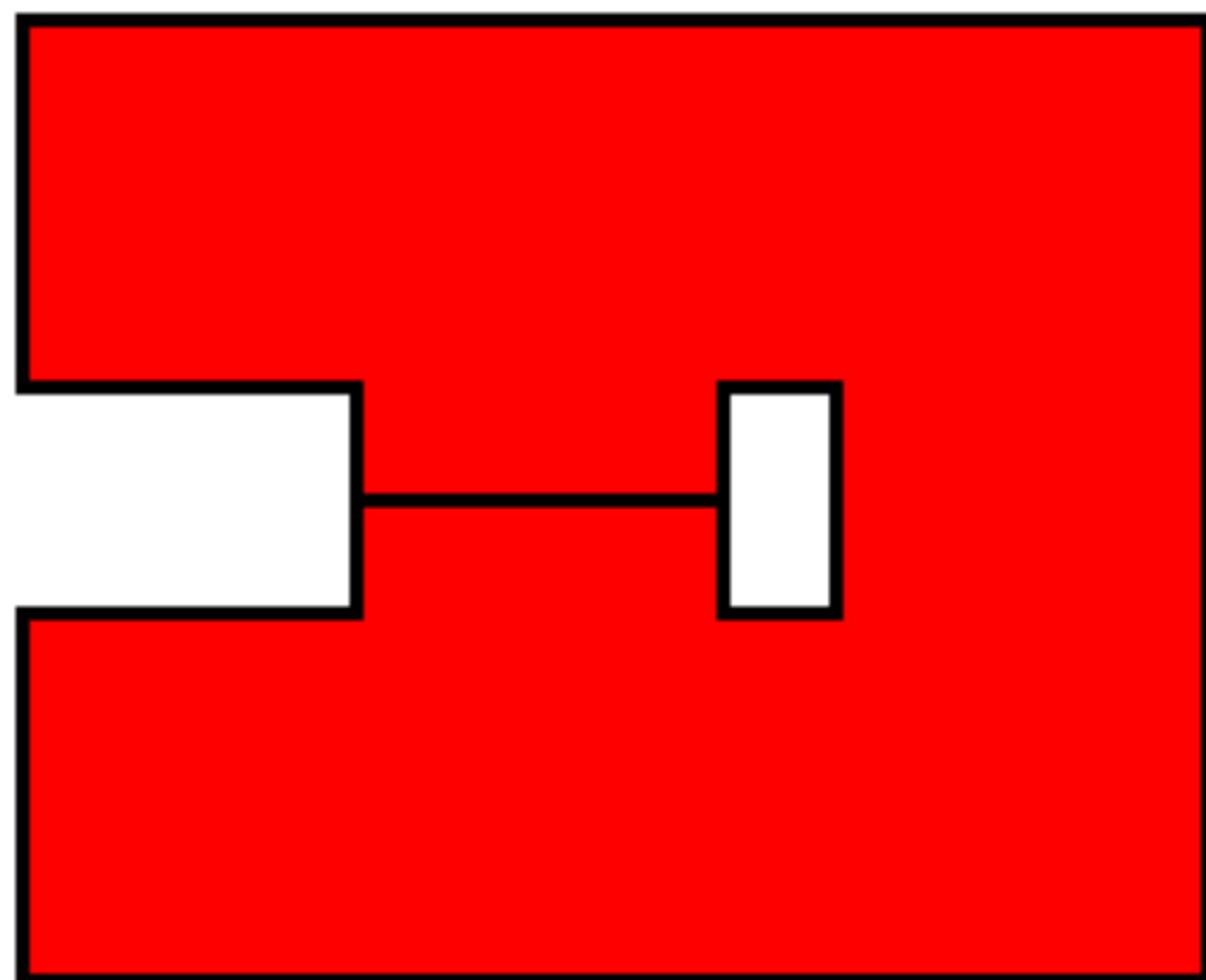


Robot



Obstacle

Room



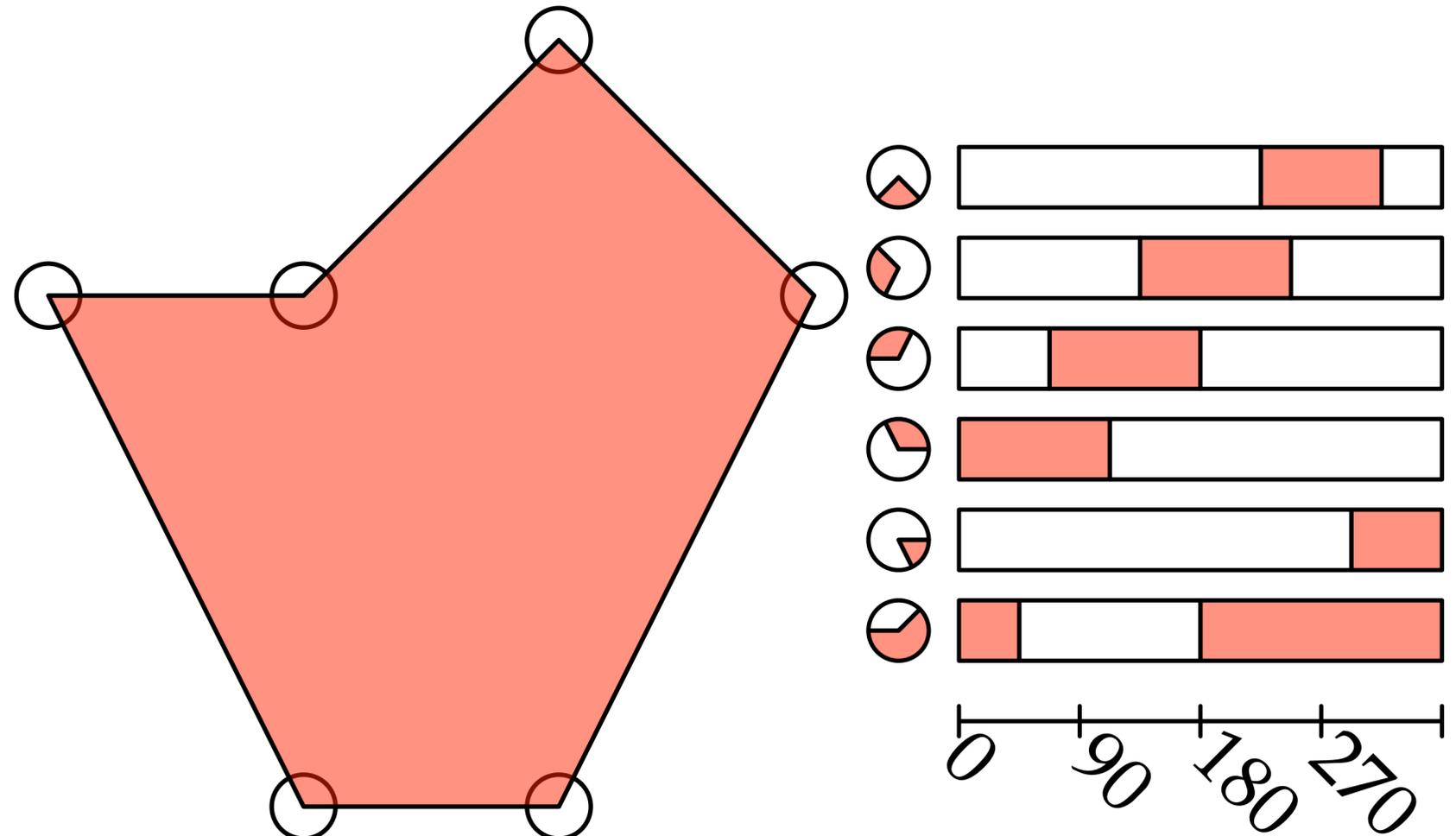
Minkowski sum

# Nef polyhedra

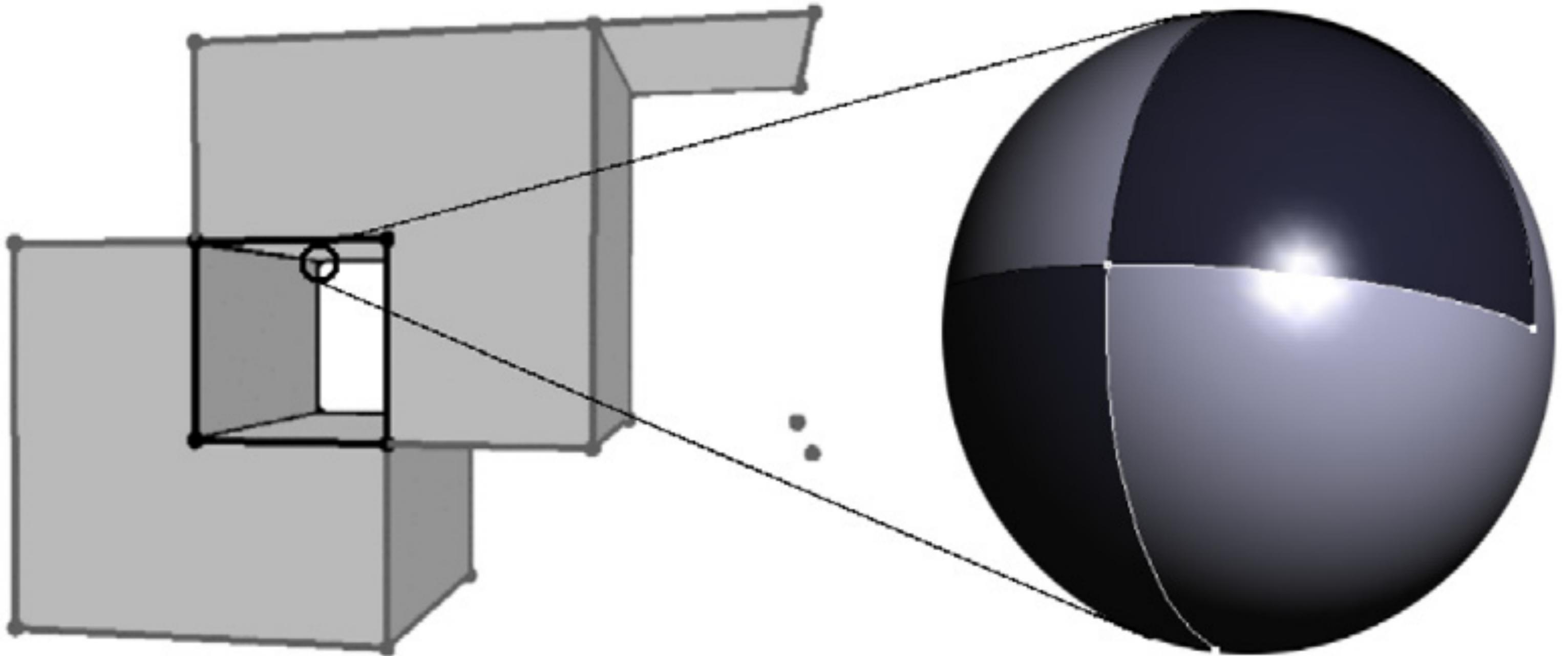
- Alternative definition of polyhedra with:
  - non-manifolds
  - robust Boolean point set operations
- Based on local pyramids
- Operations can be performed at the local pyramid level

# Local pyramids

- Intersection of an infinitesimally small circle (2D) or sphere (3D) located at each vertex
- Dimension reduction mechanism in Nef polyhedra (akin to b-rep):
  - 2D Polygon as a set of vertices + 1D ranges
  - 3D Polyhedron as a set of vertices + 2D sphere map



# Local pyramids

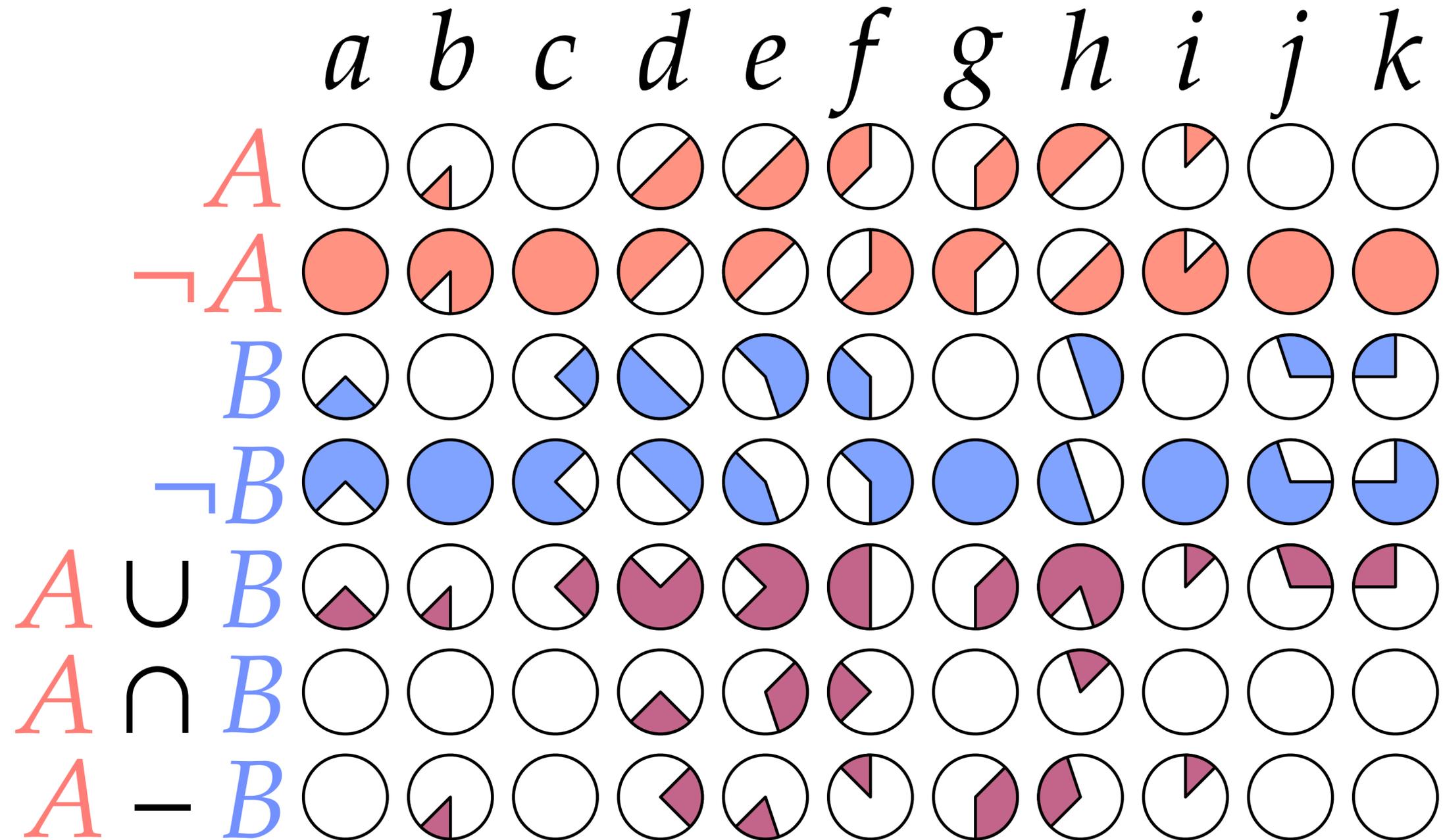
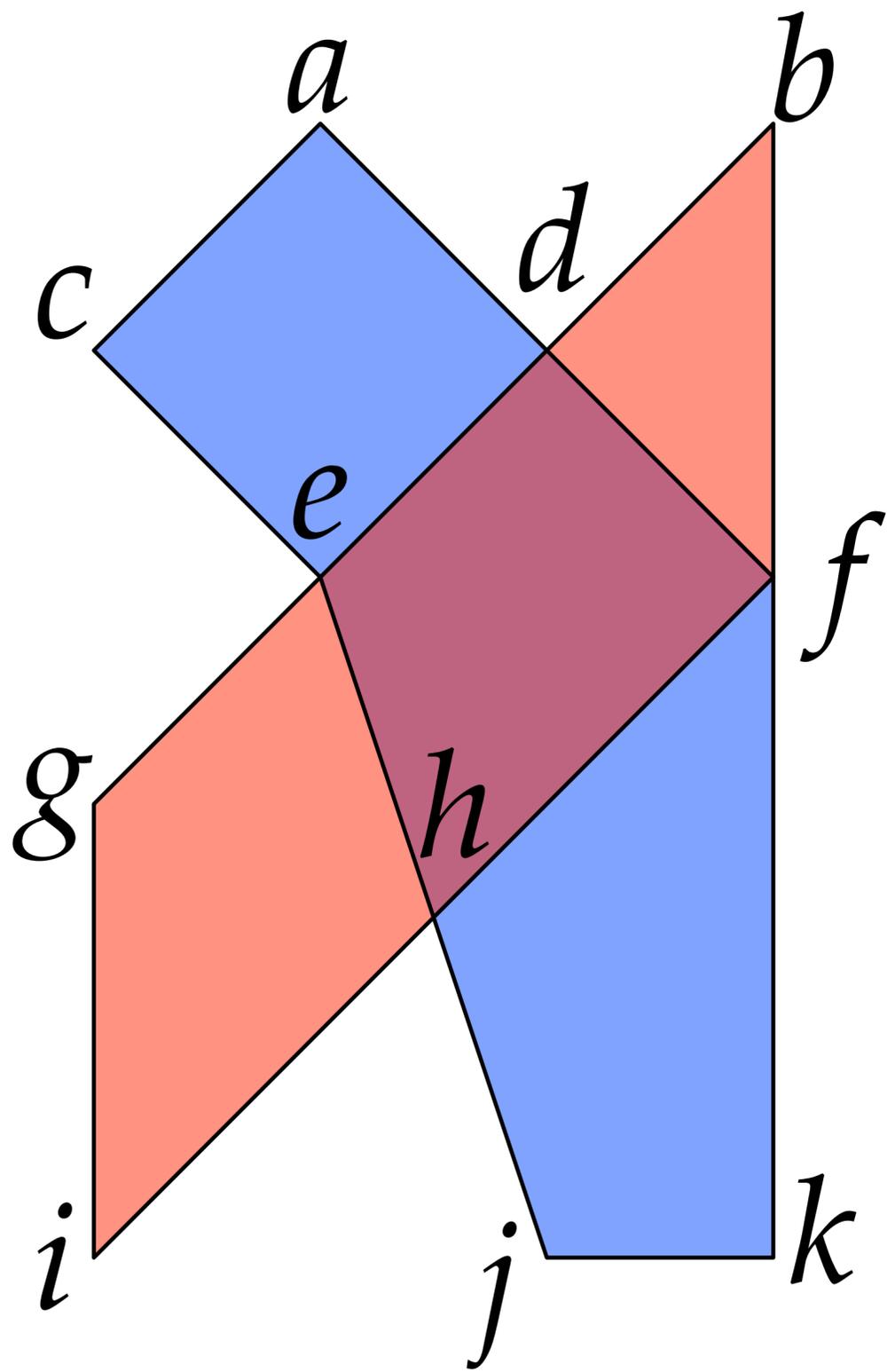




# Operations on local pyramids

1. subdivision: compute overlay -> new local pyramids
2. selection: perform operation on local pyramids
3. simplification: remove unnecessary local pyramids

# Operations on local pyramids



# What to do next?

1. Today:
  - Continue with Homework 2 (due Wednesday)
  - Go to [geo1004](#) website and study today's lesson (3D book Chapter 5)
2. Wednesday: Intro to BIM with demos and intro to Homework 3, then help with lessons or Homework 3

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# References

- Peter Hachenberger, Lutz Kettner and Kurt Mehlhorn. **Boolean operations on 3D selective Nef complexes: Data structure, algorithms, optimized implementation and experiments.** Computational Geometry 38 (1-2) pp. 64-99. 2007.
- CGAL packages: 3D Boolean Operations on Nef Polyhedra, Convex Decomposition of Polyhedra and 3D Minkowski Sum of Polyhedra.